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RESONANT SOLUTIONS AND TURNING POINTS IN AN ELLIPTIC PROBLEM WITH OSCILLATORY BOUNDARY CONDITIONS

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ABSTRACT. We consider the elliptic equation $-\Delta u + u = 0$ with nonlinear boundary conditions $\frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u)$, where the nonlinear term $\frac{g(\lambda,x,s)}{s} \to 0,$ as $|s| \to 0$ and g is oscillatory. We provide sufficient conditions on g for the existence of sequences of resonant solutions and turning points, accumulating to zero.

1. INTRODUCTION

The aim of the present work is to complement the study initiated in [Arrieta et al., 2010], and [Castro and Pardo] on the positive solutions to the following boundary-value problem:

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and sufficiently smooth domain, $N \geq 2$, λ is a real parameter, $g(\lambda, x, s) = o(s)$ as $s \to 0$ and g is oscillatory. A typical example of such a g is

$$g(x,s) := s^\alpha \left[\sin\left(\left| \frac{s}{\Phi_1(x)} \right|^\beta \right) + C \right] \qquad \text{with } \alpha + \beta > 1, \quad \beta < 0. \tag{1.2}$$

While in [Arrieta et al., 2010], [Castro and Pardo], the case $\alpha + \beta < 1$, $\beta > 0$ is treated, we focus now on the complementary range $\alpha + \beta > 1$, $\beta < 0$. The case with $\alpha < 1$ corresponds to a *bifurcation from infinity* phenomenon, see [Arrieta et al., 2007, 2009, 2010, Castro and Pardo], and [Rabinowitz, 1973]. On the contrary, the case with $\alpha > 1$ corresponds to a *bifurcation from zero* phenomenon, see [Arrieta et al., 2007] and [Crandall and Rabinowitz, 1971, Rabinowitz, 1971].

The oscillatory situation is in principle more complex than the monotone one, since order techniques such as sub and supersolutions are not applicable.

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One novelty in problem (1.1) is that the parameter appears explicitly in the boundary condition. With respect to this parameter, we perform an analysis of the local bifurcation diagram of non-negative solutions to (1.1), which turns out to be different from the case $\alpha < 1$ (see Figures 1, 2 for $\alpha > 1$ and Figure 3 for $\alpha < 1$).

Throughout this paper we assume:

(H1): $g : \mathbb{R} \times \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function (i.e. $g = g(\lambda, x, s)$ is measurable in $x \in \Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$). Moreover, there exist $G_1 \in L^r(\partial\Omega)$ with $r > N-1$, and continuous functions $\Lambda : \mathbb{R} \to \mathbb{R}^+$, and $U : \mathbb{R} \to \mathbb{R}^+$, satisfying

$$\begin{cases} \|g(\lambda, x, s)| \leq \Lambda(\lambda) G_1(x) U(s), & \forall (\lambda, x, s) \in \mathbb{R} \times \partial\Omega \times \mathbb{R}, \\ \limsup_{|s|\to 0} \dfrac{U(s)}{|s|^\alpha} < +\infty & \text{for some} \quad \alpha > 1. \end{cases}$$

(H2) **:** The partial derivative $g_s(\lambda, \cdot, \cdot) \in C(\partial\Omega \times \mathbb{R})$ (where $g_s := \frac{\partial g}{\partial s}$,) $g_s(\cdot, \cdot, 0) = 0$, and there exist $F_1 \in L^r(\partial\Omega)$, with $r > N-1$, and $\rho > 1$ such that

$$\frac{|g(\lambda, x, s) - s g_s(\lambda, x, s)|}{|s|^\rho} \leq F_1(x), \qquad \text{as } \lambda \to \sigma_1$$

for $x \in \partial\Omega$ and $s \leq \epsilon$ small enough.

Let $\{\sigma_i\}_{i=1}^\infty$ denote the sequence of *Steklov* eigenvalues of the problem

$$\begin{cases} -\Delta\Phi + \Phi &= 0, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} &= \sigma\Phi, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The Steklov eigenvalues form an increasing sequence of real numbers, $\{\sigma_i\}_{i=1}^\infty$. Each eigenvalue has finite multiplicity. The first eigenvalue σ_1 is simple and, due to Hopf's Lemma, we may assume its eigenfunction Φ_1 to be strictly positive in $\overline{\Omega}$. The eigenfunctions are orthogonal in $L^2(\partial\Omega)$, and we take $\|\Phi_1\|_{L^\infty(\partial\Omega)} = 1$.

As stated in [Arrieta et al., 2007, Theorem 8.1], due to (H1) there exists a connected set of positive solutions of (1.1) known as a *branch bifurcating from zero* . We denote it by $\mathcal{C}^+ \subset \mathbb{R} \times C(\bar{\Omega})$, and recall that for $(\lambda, u_\lambda) \in \mathcal{C}^+$

$$u = s\Phi_1 + w, \qquad \text{with } w = o(|s|) \quad \text{and} \quad |\sigma_1 - \lambda| = o(1) \quad \text{as } |s| \to 0.$$

Definition 1.1. *A solution (λ^*, u^*) of (1.1) in the branch of solutions $\mathcal{C}^+ \subset \mathbb{R} \times C(\bar{\Omega})$ is called a* **turning point** *if there is a neighborhood W of (λ^*, u^*) in $\mathbb{R} \times C(\bar{\Omega})$ such that, either $W \cap \mathcal{C}^+ \subset [\lambda^*, \infty) \times C(\bar{\Omega})$ or $W \cap \mathcal{C}^+ \subset (-\infty, \lambda^*] \times C(\bar{\Omega})$.*

Our goal is to give conditions on the nonlinear oscillatory term g that guarantee the existence of sequences accumulating to zero of *subcritical* solutions (i.e. for values of the parameter $\lambda < \sigma_1$), *supercritical* solutions (i.e. for $\lambda > \sigma_1$), *resonant* solutions (i.e. for $\lambda = \sigma_1$), and turning points.

Our main result, Theorem 1.3 below, is exemplified by the case in which g is given by (1.2). In fact we have:

Theorem 1.2. *Assume that g is given by (1.2) with $\beta < 0$. If*

$$|C| < 1, \qquad \textit{and} \quad \alpha + \beta > 1,$$

then in any neighborhood of the bifurcation point $(\sigma_1, 0)$ in $\mathbb{R} \times C(\bar{\Omega})$, the branch $\mathcal{C}^+$ of positive solutions of (1.1) contains a sequence of subcritical solutions, a sequence of supercritical solutions, a sequence of turning points, and a sequence of resonant solutions.

The proof of this Theorem follows directly from Theorem 1.3.

Theorem 1.3. *Assume the nonlinearity g satisfies hypothesis (H1) and (H2). Assume also that*

$$\left| \frac{g(\lambda, x, s) - g(\sigma_1, x, s)}{|s|^\alpha} \right| \to 0 \quad \textit{as} \quad \lambda \to \sigma_1,\ s \to 0 \tag{1.4}$$

pointwise in x.

Let $G : \mathbb{R} \times C(\bar{\Omega}) \to \mathbb{R}$ be defined by

$$G(\lambda, u) := \int_{\partial\Omega} \frac{u g(\lambda, \cdot, u)}{|u|^{1+\alpha}} \Phi_1^{1+\alpha}. \tag{1.5}$$

If there exist sequences $\{s_n\}$, $\{s'_n\}$ converging to 0^+, such that

$$\lim_{n\to+\infty} G(\sigma_1, s'_n \Phi_1) < 0 < \lim_{n\to+\infty} G(\sigma_1, s_n \Phi_1) \tag{1.6}$$

then

i) *For sufficiently large $n \gg 1$, if (λ, u) is a solution of (1.1) with*

$$P(u) := \frac{\int_{\partial\Omega} u \Phi_1}{\int_{\partial\Omega} \Phi_1^2} = s_n,$$

then (λ, u) is subcritical. Similarly, if $P(u) = s'_n$ it is supercritical. Consequently, there exist two sequences of solutions of (1.1), $\{(\lambda_n, u_n)\}$ and $\{(\lambda'_n, u'_n)\}$ converging to $(\sigma_1, 0)$ as $n \to \infty$, one of them subcritical, $\lambda_n < \sigma_1$, and the other supercritical, $\lambda'_n > \sigma_1$.

ii) *There is a sequence converging to zero of turning points $\{(\lambda^*_n, u^*_n)\}$ such that*

$$\lambda^*_n \to \sigma_1, \qquad \|u^*_n\|_{L^\infty(\partial\Omega)} \to 0, \qquad \textit{as } n \to \infty.$$

In fact, we can always choose two subsequences of turning points, one of them subcritical, $\lambda^*_{2n+1} < \sigma_1$, and the other supercritical, $\lambda^*_{2n} > \sigma_1$.

iii) *There is a sequence converging to zero of resonant solutions, i.e. there are infinitely many solutions $\{(\sigma_1, \widetilde{u}_n)\}$ of (1.1) with $\|\widetilde{u}_n\|_{L^\infty(\partial\Omega)} \to 0$.*

The behavior of positive solutions to (1.1) bifurcating from $(\sigma_1, 0)$ described in Theorems 1.2 and 1.3 is similar to that of the solutions bifurcating from (σ_1, ∞) for the sublinear problem, see [Arrieta et al., 2010] for details.

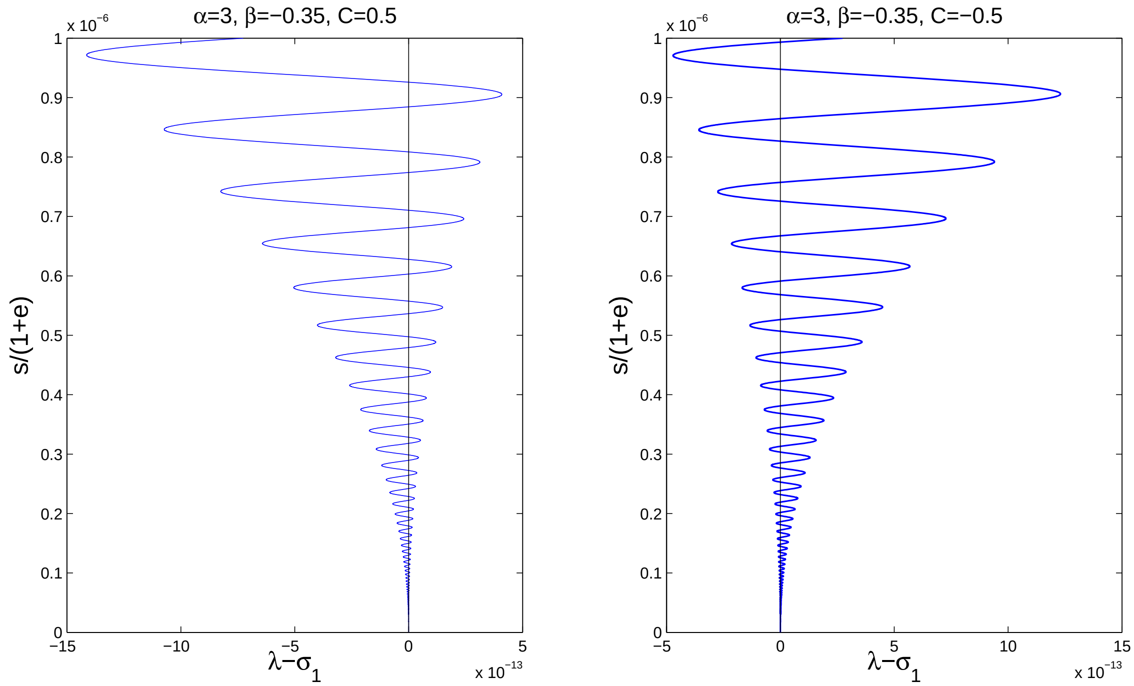

FIGURE 1. Bifurcation diagram of subcritical and supercritical solutions, containing infinitely many turning points and infinitely many resonant solutions.

The complex nature of the nonlinearity in (1.2), makes an exhaustive analysis of the global bifurcation diagram outside the scope of this work.

In [Korman, 2008] the author considers in the case $\alpha = 1$, $\beta = 1$. He assumes either $N = 1$ or Ω to be a ball and the nonlinearity to be bounded by a constant small enough. He obtains what he calls an oscillatory bifurcation. We refer the reader to [García-Melián et al., 2009] for related problems with nonlinear boundary conditions.

This paper is organized as follows. Section 2 contains the proof of our main result, giving sufficient conditions for having subcritical, supercritical, and resonant solutions. Section 3 presents two examples; explicit resonant solutions for the oscillatory nonlinearity (1.2) and the one dimensional case.

2. SUBCRITICAL, SUPERCRITICAL AND RESONANT SOLUTIONS

In this section we give sufficient conditions for the existence of a branch of solutions to (1.1) bifurcating from zero which is neither *subcritical* ($\lambda < \sigma_1$), nor *supercritical*, ($\lambda < \sigma_1$). From this, we conclude the existence of infinitely many *turning points*, see Definition 1.1, and an infinite number of solutions for the resonant problem, i.e. for $\lambda = \sigma_1$. This is achieved in Theorem 1.3

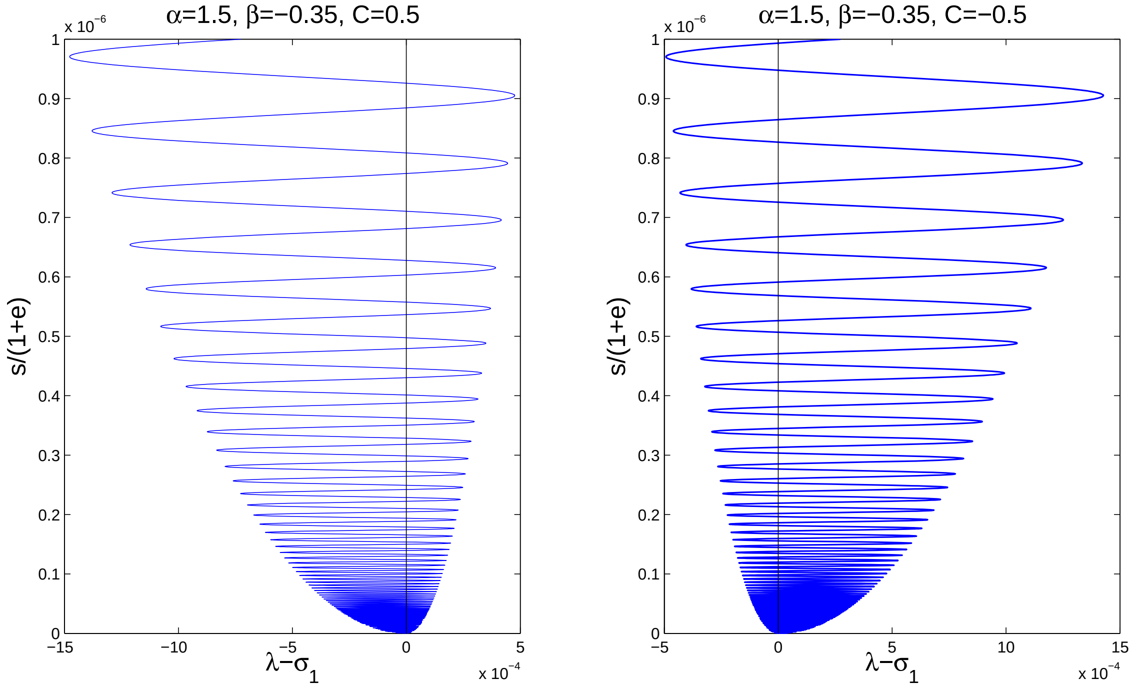

FIGURE 2. A bifurcation diagram for $\alpha = 1.5$..

At this step, we analyze when the parameter may cross the first Steklov eigenvalue. To do that, we look at the asymptotic rate of the nonlinear term

$$\underline{\mathbf{G}}_{0+} := \int_{\partial\Omega} \liminf_{(\lambda,s)\to(\sigma_1,0)} \frac{sg(\lambda,\cdot,s)}{|s|^{1+\alpha}} \Phi_1^{1+\alpha} \tag{2.1}$$

for $\alpha > 1$. Changing $\liminf$ by $\limsup$ we define the number $\overline{\mathbf{G}}_{0+}$. If $\underline{\mathbf{G}}_{0+} > 0$, then $\mathcal{C}^+$ is subcritical, and if $\overline{\mathbf{G}}_{0+} < 0$, then $\mathcal{C}^+$ is supercritical in a neighborhood of $(\sigma_1, 0)$. See [Arrieta et al., 2009, Theorems 3.4and 3.5] for the bifurcation from infinity case. In this paper we consider nonlinearities for which

$$\underline{\mathbf{G}}_{0+} < 0 < \overline{\mathbf{G}}_{0+}.$$

We shall argue as in [Arrieta et al., 2010] for the bifurcation from infinity case. To determine whether a sequence of solutions (λ_n, u_n) is subcritical or supercritical, one must check the sign of

$$\liminf_{n\to\infty} G(\lambda_n, u_n) \qquad \text{and} \qquad \limsup_{n\to\infty} G(\lambda_n, u_n), \tag{2.2}$$

where G is defined by (1.5). This is done in Lemma 2.3.

In Proposition 2.2, it is proved that when g is such that

$$|g(\lambda, x, s)| = O\left(|s|^{\alpha}\right) \text{ as } |s| \to 0 \text{ for some } \alpha > 1,$$

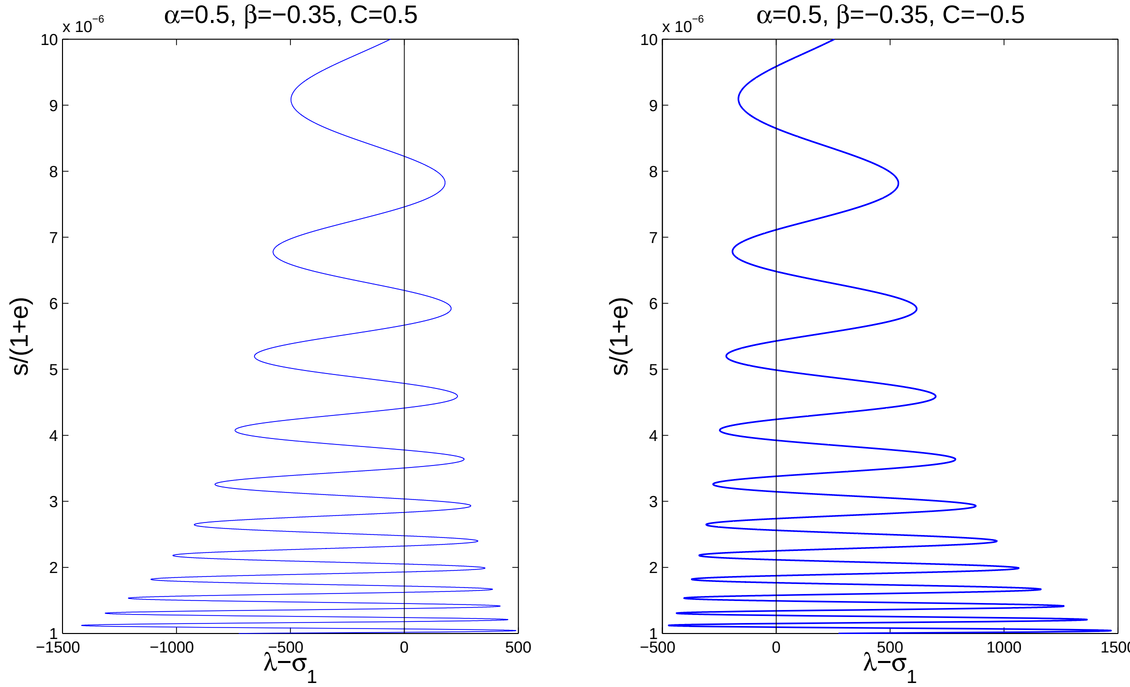

FIGURE 3. A bifurcation diagram for $\alpha = 0.5$

then the solutions in $\mathcal{C}^{\pm}$ can be described as

$$u_n = s_n \Phi_1 + w_n, \qquad \text{where} \quad \int_{\partial\Omega} w_n \Phi_1 = 0 \quad \text{and} \qquad w_n = O(|s_n|^{\alpha}) \text{ as } n \to 0.$$

We unveil the signs of the expressions in (2.2) by just looking at the signs of the expressions in (2.2) at $\lambda_n = \sigma_1$ and $u_n = s_n \Phi_1$. This is achieved in Lemma 2.4.

For this we first consider a family of linear Steklov problems with a variable nonhomogeneous term at the boundary h depending on the parameter λ

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = \lambda u + h(\lambda, x), & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

where $h(\lambda, \cdot) \in L^r(\partial\Omega)$, $r > N-1$ and $\lambda \in (-\infty, \sigma_2)$.

We use the decomposition

$$L^r(\partial\Omega) = \text{span}[\Phi_1] \oplus \text{span}[\Phi_1]^{\perp}, \quad \text{where} \quad \text{span}[\Phi_1]^{\perp} := \left\{ u \in L^r(\partial\Omega) : \int_{\partial\Omega} u\, \Phi_1 = 0 \right\}.$$

For $h(\lambda, \cdot) \in L^r(\partial\Omega)$, with $r > N-1$, we write

$$h(\lambda, \cdot) = a_1(\lambda)\Phi_1 + h_1(\lambda, \cdot), \quad \text{with } a_1(\lambda) = \frac{\int_{\partial\Omega} h(\lambda, \cdot)\, \Phi_1}{\int_{\partial\Omega} {\Phi_1}^2}, \quad \int_{\partial\Omega} h_1(\lambda, \cdot)\Phi_1 = 0. \tag{2.4}$$

For $\lambda \neq \sigma_1$ the solution $u = u(\lambda)$ of (2.3) has a unique decomposition

$$u = \frac{a_1(\lambda)}{\sigma_1 - \lambda}\,\Phi_1 + w, \quad \text{where} \quad \int_{\partial\Omega} w\Phi_1 = 0, \tag{2.5}$$

and $w = w(\lambda) \in \text{span}[\Phi_1]^\perp$ solves the problem

$$\begin{cases} -\Delta w + w &= 0, & \text{in } \Omega \\ \frac{\partial w}{\partial n} &= \lambda w + h_1(\lambda, x), & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Note that in (2.6) $w(\lambda) \in \text{span}[\Phi_1]^\perp$ is also well defined for $\lambda = \sigma_1$. Moreover, we have:

Lemma 2.1. *For each compact set $K \subset (-\infty, \sigma_2) \subset \mathbb{R}$ there exists a constant $C = C(K)$, independent of λ, such that*

$$\|w(\lambda)\|_{L^\infty(\partial\Omega)} \leq C\|h_1(\lambda, \cdot)\|_{L^r(\partial\Omega)} \quad \textit{for any} \quad \lambda \in K,$$

where $w \in$ span$[\Phi_1]^\perp$ is the solution of (2.6) and $h_1 \in$ span$[\Phi_1]^\perp$ is defined in (2.4).

Proof. See Lemma 3.1 of [Arrieta et al., 2010]. □

Now we turn our attention to the nonlinear problem (1.1). Recall that for solutions (λ, u) close to the bifurcation point $(\sigma_1, 0)$ we have

$$u = s\Phi_1 + w, \qquad \text{where } w = o(s), \quad w \in \text{span}[\Phi_1]^\perp \qquad \text{as } s \to 0. \tag{2.7}$$

We define

$$P(u) := \frac{\int_{\partial\Omega} u(\cdot)\,\Phi_1}{\int_{\partial\Omega} \Phi_1^2}. \tag{2.8}$$

Next, we give sufficient conditions on the nonlinear term g in (1.1), for $w = O(|s|^\alpha)$ as $s \to 0$, see (2.7). We restrict ourselves below to the branch of positive solutions; a completely analogous result holds for the branch of negative solutions. The following Proposition is essentially Proposition 3.2 in [Arrieta et al., 2010] rewritten for $s \to 0$; we include the proof by the sake of completeness.

Proposition 2.2. *Assume g satisfies hypotheses (H1), (H2) and that for some $\alpha > 1$ and $\varepsilon > 0$ there exists a function G_1 such that for $|\lambda - \sigma_1| < \varepsilon$, and $s \in (0, \varepsilon)$ and $x \in \partial\Omega$ we have*

$$\frac{|g(\lambda, x, s)|}{|s|^\alpha} \leq G_1(x), \quad G_1 \in L^r(\partial\Omega), \quad r > N - 1. \tag{2.9}$$

Then, there exists an open set $\mathcal{O} \subset \mathbb{R} \times C(\bar{\Omega})$ of the form $\mathcal{O} = \{(\lambda, u) : |\lambda - \sigma_1| < \delta_0, \|u\|_{L^\infty(\Omega)} < s_0\}$ for some δ_0 and s_0, such that

(i) There exists a constant C_1 independent of λ such that, if $(\lambda, u) \in \mathcal{C}^+ \cap \mathcal{O}$ and $(\lambda, u) \neq (\sigma_1, 0)$ then $u = s\Phi_1 + w$, where $s > 0$, $w \in$ span$[\Phi_1]^\perp$ and

$$\|w\|_{L^\infty(\partial\Omega)} \leq C_1\|G_1\|_{L^r(\partial\Omega)}\,|s|^\alpha, \qquad \textit{as } |s| \to 0$$

(ii) There exists a constant $S_0 > 0$ such that for all $|s| \leq S_0$ there exists $(\lambda, u) \in \mathcal{C}^+ \cap \mathcal{O}$ satisfying $u = s\Phi_1 + w$, with $w \in$ span$[\Phi_1]^\perp$.

(iii) Moreover, for any $(\lambda, u) \in \mathcal{C}^+ \cap \mathcal{O}$, $u = s\Phi_1 + w$, *with* $w \in$ *span*$[\Phi_1]^\perp$,

$$|\sigma_1 - \lambda| \leq C_2 |s|^{\alpha - 1}, \qquad \textit{as } |s| \to 0,$$

with C_2 *independent of* λ, *in fact*

$$C_2 = \frac{2\|G_1\|_{L^1(\partial\Omega)}}{\int_{\partial\Omega} \Phi_1^2}.$$

Proof. Note that (2.9), Lemma 2.1, and the fact that, from (2.7), $\Phi_1 + w/s \to \Phi_1$ as $s \to 0$ in $L^\infty(\partial\Omega)$ imply that $\|w\|_{L^\infty(\partial\Omega)} \leq C|s|^\alpha$ as $s \to 0$. This proves part i).

To prove part ii) note that $\mathcal{C}^+ \cap \mathcal{O}$ is connected. Hence, using the decomposition in (2.7), we have $u = s\Phi_1 + w$ with $w \in$ span$[\Phi_1]^\perp$. Since the projection P is continuous, see (2.8), the set $\{s \in \mathbb{R} :$ (1.1) has a solution of the form $u = s\Phi_1 + w$ and $w \in \left[\text{span}[\Phi_1]^\perp\right]\}$ contains an interval in $\mathbb{R}$ containing zero.

To prove part iii) we observe that if (λ, u) is a solution of (1.1), $u = s\Phi_1 + w$, with $w \in$ span$[\Phi_1]^\perp$, multiplying the equation by the first Steklov eigenfunction $\Phi_1 > 0$ and integrating by parts we obtain,

$$(\sigma_1 - \lambda) s \int_{\partial\Omega} \Phi_1^2 = \int_{\partial\Omega} g(\lambda, x, s\Phi_1 + w)\Phi_1.$$

Taking into account that

$$\frac{|g(\lambda, x, s\Phi_1 + w)|}{|s|} = \frac{|g(\lambda, x, s\Phi_1 + w)|}{|s\Phi_1 + w|} \left|\Phi_1 + \frac{w}{s}\right| \to 0, \qquad \text{as} \qquad s \to 0$$

we get $\lambda \to \sigma_1$ as $s \to 0$.

Moreover, from (2.9), we obtain that

$$|g(\lambda, x, s\Phi_1 + w)| = |s|^\alpha \frac{|g(\lambda, x, s\Phi_1 + w)|}{|s\Phi_1 + w|^\alpha} \left|\Phi_1 + \frac{w}{s}\right|^\alpha \leq |s|^\alpha G_1(x) \left|\Phi_1 + \frac{w}{s}\right|^\alpha,$$

and therefore

$$|\sigma_1 - \lambda| \leq \frac{|s|^{\alpha-1}}{\int_{\partial\Omega} \Phi_1^2} \int_{\partial\Omega} G_1(x) \left|\Phi_1 + \frac{w}{s}\right|^\alpha \Phi_1 \leq C\|G_1\|_{L^r(\partial\Omega)} |s|^{\alpha-1}$$

which ends the proof. □

Our next Lemma is essentially Lemma 3.1 in [Arrieta et al., 2009] rewritten for $s \to 0$. We omit its proof. . It allows us to estimate $\sigma_1 - \lambda_n$ as λ_n converges σ_1.

Lemma 2.3. *Assume the nonlinearity* g *satisfies hypotheses* (H1) *and* (H2).

Let (λ_n, u_n) *be a sequence of solutions of* (1.1) *with* $\lambda_n \to \sigma_1$ *and* $\|u_n\|_{L^\infty(\partial\Omega)} \to 0$. *If* $u_n > 0$, *then*

$$\begin{aligned}\frac{\underline{\mathbf{G}}_{0^+}}{\int_{\partial\Omega}\Phi_1{}^2} &\leq \frac{1}{\int_{\partial\Omega}\Phi_1{}^2}\liminf_{n\to\infty} G(\lambda_n,u_n)\\ &\leq \liminf_{n\to\infty}\frac{\sigma_1-\lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}} \leq \limsup_{n\to\infty}\frac{\sigma_1-\lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}}\\ &\leq \frac{1}{\int_{\partial\Omega}\Phi_1{}^2}\limsup_{n\to\infty} G(\lambda_n,u_n) \leq \frac{\overline{\mathbf{G}}_{0^+}}{\int_{\partial\Omega}\Phi_1{}^2}\end{aligned}$$

A similar statement is obtained for the case $u_n<0$*, just changing* $\underline{\mathbf{G}}_{0^+}$ *by* $\underline{\mathbf{G}}_{0^-}$ *and* $\overline{\mathbf{G}}_{0^+}$ *by* $\overline{\mathbf{G}}_{0^-}$.

Let $\{s_n\}$ and $\{s'_n\}$ satisfy

(2.10) $$-\infty < \lim_{n\to+\infty} G(\sigma_1, s'_n\Phi_1) < 0 < \lim_{n\to+\infty} G(\sigma_1, s_n\Phi_1) < \infty.$$

In order to prove Theorem 1.3, we show that the signs in (2.2) can be deduced from those of (2.10). This is stated in the following result, which is a slight variation of [Arrieta et al., 2010, Lemma 3.3]

Lemma 2.4. *Assume that* g *satisfies hypotheses* (H1), (H2) *and* (1.4).

If $(\lambda_n,s_n)\to(\sigma_1,0)$ *and there exists a constant* C *such that* $\|w_n\|_{L^\infty(\partial\Omega)}\leq C|s_n|^\alpha$ *for all* $n\to 0$*, then*

$$\liminf_{n\to+\infty} G(\lambda_n, s_n\Phi_1+w_n) \geq \liminf_{n\to+\infty} G(\sigma_1, s_n\Phi_1),$$

where G *is given by* (1.5)*. Similarly*

$$\limsup_{n\to+\infty} G(\lambda_n, s_n\Phi_1+w_n) \leq \limsup_{n\to+\infty} G(\sigma_1, s_n\Phi_1).$$

Proof. Throughout this proof, C denotes several constants depending only on (Ω,g). Given $\varepsilon>0$, assume that $|(\lambda_n,s_n)-(\sigma_1,0)|<\varepsilon$.

By the mean value theorem we have

$$\begin{aligned} g(\lambda_n,x,s_n\Phi_1+w_n)-g(\lambda_n,x,s_n\Phi_1) &= w_n\int_0^1 g_s(\lambda_n,\cdot,s_n\Phi_1+\tau w_n)\,d\tau\\ &\leq \|w_n\|_{L^\infty(\partial\Omega)}\sup_{\tau\in[0,1]}\|g_s(\lambda_n,\cdot,s_n\Phi_1+\tau w_n)\|_{L^\infty(\partial\Omega)}\,. \end{aligned} \tag{2.11}$$

Therefore

$$\int_{\partial\Omega} \Big| g(\lambda_n, x, s_n\Phi_1 + w_n) - g(\lambda_n, x, s_n\Phi_1)\Big| \, \Phi_1 \, dx$$
$$\le \|w_n\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \sup_{\tau\in[0,1]} \|g_s(\lambda_n, \cdot, s_n\Phi_1 + \tau w_n)\|_{L^\infty(\partial\Omega)}$$
$$\le |\partial\Omega| \|w_n\|_{L^\infty(\partial\Omega)} \sup_{\tau\in[0,1]} \|g_s(\lambda_n, \cdot, s_n\Phi_1 + \tau w_n)\|_{L^\infty(\partial\Omega)} . \tag{2.12}$$

By hypotheses (H1)-(H2), for all $x \in \partial\Omega$,

$$\frac{|g_s(\lambda_n, x, s)|}{|s|^{\gamma-1}} \le |s|^{\rho-\gamma} F_1(x) + C|s|^{\alpha-\gamma} G_1(x) \max\{\Lambda(\lambda_n),\ n \ge 1\} =: D_1(x), \tag{2.13}$$

for n large, and $\gamma = \min\{\rho, \alpha\} > 1$. Hence, $D_1 \in L^r(\partial\Omega)$ with $r > N-1$, and

$$\sup_{|s|\le 1/n} |g_s(\lambda_n, x, s)| \le D_1(x) \left(\frac{1}{n}\right)^{\gamma-1} \qquad \text{with} \quad \gamma > 1. \tag{2.14}$$

Since $\|w_n\|_{L^\infty(\partial\Omega)} = O(|s_n|^\alpha)$, from (2.12) and (2.14)

$$\int_{\partial\Omega} \frac{|g(\lambda_n, \cdot, s_n\Phi_1 + w_n) - g(\lambda_n, \cdot, s_n\Phi_1)|}{|s_n|^\alpha} \Phi_1 \le C \sup_{\tau\in[0,1]} \|g_s(\lambda_n, \cdot, s_n\Phi_1 + \tau w_n)\|_{L^\infty(\partial\Omega)}$$
$$\le C \sup_{|s|\le 1/n} \|g_s(\lambda_n, \cdot, s)\|_{L^\infty(\partial\Omega)} \longrightarrow 0 \tag{2.15}$$

as $n \to \infty$.

Therefore

$$\begin{aligned}
\liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n g(\lambda_n, \cdot, s_n\Phi_1 + w_n)}{|s_n|^{1+\alpha}} \Phi_1 \ge & \\
\ge \lim_{n\to\infty} & \int_{\partial\Omega} \frac{s_n g(\lambda_n, \cdot, s_n\Phi_1 + w_n) - s_n g(\lambda_n, \cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1 \\
& + \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n g(\lambda_n, \cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1 \\
= \liminf_{n\to+\infty} & \int_{\partial\Omega} \frac{s_n g(\lambda_n, \cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1 \\
= \liminf_{n\to+\infty} & \int_{\partial\Omega} \frac{s_n g(\sigma_1, \cdot, s_n\Phi_1)}{|s_n|^{1+\alpha}} \Phi_1,
\end{aligned}$$

where we used (2.15) and (1.4) respectively.

Now note that, multiplying and dividing by $\left|\Phi_1 + \frac{w_n}{s_n}\right|^\alpha$ the integrand of the left hand side above can be written as

$$\frac{s_n g(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n|^{1+\alpha}}\,\Phi_1=\frac{(s_n\Phi_1+w_n)g(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n\Phi_1+w_n|^{1+\alpha}}\left|\Phi_1+\frac{w_n}{s_n}\right|^{\alpha}\Phi_1.$$

Then, (H2) and the fact that $\Phi_1+w_n/s_n\to\Phi_1$ in $L^\infty(\partial\Omega)$ concludes the proof. □

Now we prove the first main result in this paper. Roughly speaking, it states that if there are a sequence of subcritical solutions and another of supercritical solutions, since the solution set is connected, there are infinitely many turning points and infinitely many resonant solutions. We prove the result for the positive branch. The same conclusions can be attained for the connected branch of negative solutions bifurcating from zero.

Proof of Theorem 1.3 From Proposition 2.2, ii), consider any two sequences of solutions of (1.1), such that $(\lambda_n,u_n)\to(\sigma_1,0)$ and $(\lambda_n',u_n')\to(\sigma_1,0)$ in $\mathcal{C}^+$ with

$$P(u_n)=\frac{\int_{\partial\Omega}u_n\Phi_1}{\int_{\partial\Omega}\Phi_1^2}=s_n,\quad\text{and}\quad P(u_n')=\frac{\int_{\partial\Omega}u_n'\Phi_1}{\int_{\partial\Omega}\Phi_1^2}=s_n'.$$

Writing $u_n=s_n\Phi_1+w_n$, with $w_n\in\text{span}[\Phi_1]^\perp$, from Proposition 2.2 i), we have $\|w_n\|_{L^\infty(\partial\Omega)}=O(|s_n|^\alpha)$. From Lemmata 2.3, and 2.4, hypotheses (1.4) and (1.6) we get that

$$\begin{aligned}\liminf_{n\to\infty}\frac{\sigma_1-\lambda_n}{\|u_n\|^{\alpha-1}_{L^\infty(\partial\Omega)}}&\geq\frac{1}{\int_{\partial\Omega}\Phi_1^2}\liminf_{n\to\infty}\int_{\partial\Omega}\frac{(s_n\Phi_1+w_n)g(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n\Phi_1+w_n|^{1+\alpha}}\Phi_1^{1+\alpha}\\&\geq\frac{1}{\int_{\partial\Omega}\Phi_1^2}\liminf_{n\to+\infty}\int_{\partial\Omega}\frac{s_n g(\lambda_n,\cdot,s_n\Phi_1)}{|s_n|^{1+\alpha}}\Phi_1\\&=\frac{1}{\int_{\partial\Omega}\Phi_1^2}\liminf_{n\to+\infty}\int_{\partial\Omega}\frac{s_n g(\sigma_1,\cdot,s_n\Phi_1)}{|s_n|^{1+\alpha}}\Phi_1\ >0,\end{aligned}$$

and therefore $\lambda_n<\sigma_1$.

Analogously, for (λ_n',u_n') we get $\lambda_n'>\sigma_1$. Hence i) is proved.

To prove ii), assume, by choosing subsequences if necessary, that $s_n>s_n'>s_{n+1}$ for all $n\geq0$ and that $0<s_n,s_n'\leq S_0$ where S_0 is the one from Proposition 2.2, part ii). In particular, from i) and ii) of Proposition 2.2 we have that if $(\lambda,u)\in\mathcal{C}^+$ and $P(u)=s<S_0$ then $\|u\|_{L^\infty(\partial\Omega)}\leq(1+C_1\|G_1\|_{L^r(\partial\Omega)}|S_0|^{\alpha-1})s$. Taking S_0 small enough we may assume that $\|u\|_{L^\infty(\partial\Omega)}\leq2s$.

Let

$$K_n=\{(\lambda,u)\in\mathcal{C}^+,\ \text{with}\,P(u)=s,\ \text{and}\ s_n\geq s\geq s_{n+1}\}\tag{2.16}$$

Let us see that, for each $n\in\mathbb{N}$, K_n is a compact subset of $\mathbb{R}\times C(\bar\Omega)$. Let $\{(\mu_k,v_k)\}$ be a sequence in K_n. Without loss of generality we may assume that $\{\mu_k\}$ converges to μ^*. Since $v_k=t_k\Phi_1+w_k$ with $w_k=O(|t_k|^\alpha)$ and $s_n\geq t_k=:P(v_k)\geq s_{n+1}$, for all k, then $\|v_k\|_{C(\partial\Omega)}\leq t_k+\|w_k\|_{C(\partial\Omega)}\leq C$ with C independent of k. This and Proposition 2.3 of [Arrieta et al., 2007] we have

$$\|v_k\|_{C(\bar\Omega)}\leq C_!(1+\|v_k\|_{C(\partial\Omega)})\leq C,\tag{2.17}$$

where, again, C is independent of k. Since the embedding $C^{\gamma}(\bar{\Omega}) \to C^{\gamma'}(\bar{\Omega})$ is compact for $0 < \gamma' < \gamma$ we may further assume that the sequence $\{v_k\}$ converges to some $u^* \in C^{\gamma'}(\bar{\Omega})$. This, hypothesis $(H1)$ and the dominated convergence theorem imply that $\{g(\mu_k, \cdot, v_k)\}$ converges to $g(\mu^*, \cdot, u^*)$ in $L^r(\partial\Omega)$. Therefore, since

$$
\begin{aligned}
-\Delta v_k + v_k = 0 \quad & in\ \Omega \\
\frac{\partial v_k}{\partial n} = \mu_k v_k + g(\mu_k, x, v_k) \quad & on\ \partial\Omega,
\end{aligned}
\tag{2.18}
$$

passing to the limit in the weak sense we have

$$
\begin{aligned}
-\Delta u^* + u^* = 0 \quad & in\ \Omega \\
\frac{\partial u^*}{\partial n} = \mu^* u^* + g(\lambda^*, x, u^*) \quad & on\ \partial\Omega.
\end{aligned}
\tag{2.19}
$$

By the continuity of the projection operator we also have $s_n \geq s^* = P(u*) = \lim_{k\to\infty} P(v_k) \geq s_{n+1}$. Hence $(\mu^*, u^*) \in K_n$, which proves that K_n is compact.

Since $s_n > s'_n > s_{n+1}$, there exists $(\lambda, u) \in K_n$ with $\lambda > \sigma_1$. Hence, if we define

$$
\lambda_n^* = \sup\{\lambda : (\lambda, u) \in K_n\},
\tag{2.20}
$$

then $\lambda_n^* \geq \lambda'_n > \sigma_1$, see part i). From the compactness of K_n there exists u_n^* such that $(\lambda_n^*, u_n^*) \in K_n$. From the definition of λ_n^*, if (λ, u) is a solution of (1.1) with $s_n > P(u_n) > s_{n+1}$, then $\lambda \leq \lambda_n^*$ which proves that (λ_n^*, u_n^*) is a (supercritical) turning point.

With a completely symmetric argument, using the sets

$$
K'_n = \{(\lambda, u) \in \mathcal{C}^+, \text{ with } P(u) = s, \text{ and } s'_n \geq s \geq s'_{n+1}\}
$$

and defining $\lambda_n'^* = \inf\{\lambda : (\lambda, u) \in K'_n\}$ we show the existence of u_* such that $(\lambda_n'^*, u_n'^*) \in K'_n$ is a (subcritical) turning point.

In order to prove the existence of resonant solutions, let us show now the following: there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ both sets K_n and K'_n contain resonant solutions, that is, solutions of the form (σ_1, u).

Let us use a *reductio ad absurdum* argument for the sets K_n. If this is not the case, then there will exist a sequence of integers numbers $n_j \to +\infty$ such that K_{n_j} does not contain any resonant solution. This implies that the compact sets $K_{n_j}^+ = \{(\lambda, u) \in K_{n_j} : \lambda \geq \sigma_1\}$ can be written as $K_{n_j}^+ = \mathcal{C}^+ \cap \{(\lambda, u) \in \mathbb{R} \times C(\partial\Omega) : \lambda > \sigma_1, s_{n_j} > P(u) > s_{n_j+1}\}$ and therefore $K_{n_j}^+$ contains at least a connected component of $\mathcal{C}^+$. Moreover it is nonempty since we know that there exists at least one solution (λ, u) with $P(u) = s'_{n_j} \in (s_{n_j}, s_{n_j+1})$ and therefore $\lambda > \sigma_1$. The fact that we can construct a sequence of connected components of $\mathcal{C}^+$ contradicts the fact that $\mathcal{C}^+$ is a connected near $(\sigma_1, 0) \in \mathbb{R} \times C(\bar{\Omega})$.

A completely symmetric argument can be applied to the sets K'_n.□

3. Two examples

3.1. Resonant solutions for the oscillatory nonlinearity. In [Arrieta et al., 2007, Theorem 8.1] it is proved that if $\alpha > 1$, for any $\beta \in \mathbb{R}$, and $C \in \mathbb{R}$, there is an unbounded branch of positive solutions. Assume from now that $\beta < 0$.

Taking , we see that

$$u_k(x) := [\text{asin}(-C) + k\pi]^{1/\beta} \Phi_1(x), \qquad k \geq 0,$$

defines a sequence of resonant solutions to (1.1).

3.2. A one dimensional example. Now we consider the onedimensional version of (1.1), where most computations can be made explicit.

Observe that equation (1.1) in the one dimensional domain $\Omega = (0,1)$ reads

$$\begin{cases} -u_{xx} + u &= 0, & \text{in } (0,1) \\ -u_x(0) &= \lambda u + g(\lambda, 0, u(0)). \\ u_x(1) &= \lambda u + g(\lambda, 1, u(1)), \end{cases}$$

The general solution of the differential equation is $u(x) = ae^x + be^{-x}$ and therefore the nonlinear boundary conditions provide two nonlinear Eqs. in terms of two constants a and b. The function $u = ae^x + be^{-x}$ is a solution if (λ, a, b) satisfy

$$\begin{pmatrix} -(1+\lambda) & (1-\lambda) \\ (1-\lambda)e & -(1+\lambda)e^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} =$$

$$\begin{pmatrix} g(\lambda, 0, a+b) \\ g(\lambda, 1, ae + be^{-1}) \end{pmatrix}$$

In this case we only have two Steklov eigenvalues,

$$\sigma_1 = \frac{e-1}{e+1} < \sigma_2 = \frac{1}{\sigma_1} = \frac{e+1}{e-1}.$$

Choosing $g(\lambda, s) = g(s)$, and restricting the analysis to symmetric solutions $u_s(x) = s(e^x + e^{1-x})$, with $s \in \mathbb{R}$, it is easy to prove that $u_s(x)$ is a solution if and only if λ satisfies

$$\lambda(s) = \sigma_1 - \frac{g(s(e+1))}{s(e+1)}, \qquad s > 0. \tag{3.1}$$

Therefore, whenever $g(u) = o(u)$ at zero, there is a branch of solutions $(\lambda(s), u_s) \to (\sigma_1, 0)$ as $s \to 0$.

Fix now

$$g(s) = s^\alpha \sin(s^\beta) \qquad \text{for any } \alpha > 1,\ \beta < 0.$$

From definition (2.1) we can write

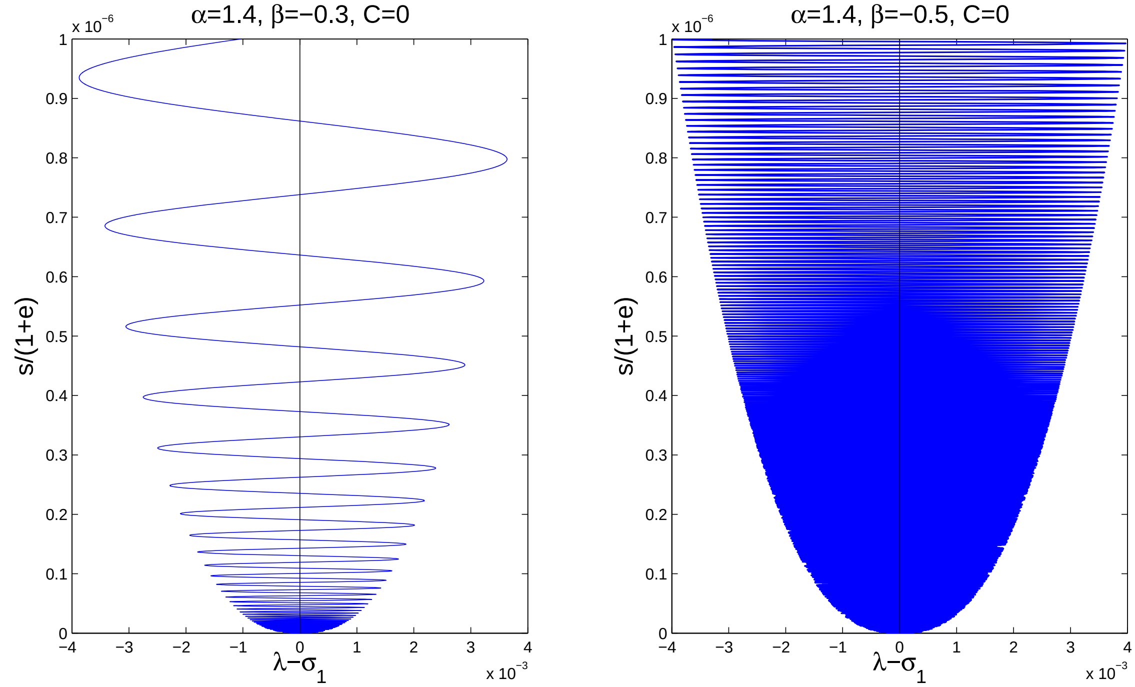

FIGURE 4. $\alpha = 1.4$, $\beta = -0.3$ and $\beta = -0.5$.

$$\underline{\mathbf{G}}_{0^+} := \int_{\partial\Omega} \liminf_{s\to 0^+} \frac{sg(s)}{|s|^{1+\alpha}}\, \Phi^{1+\alpha} = \int_{\partial\Omega} \liminf_{s\to 0^+} \sin(s^\beta)\, \Phi^{1+\alpha} = -\int_{\partial\Omega} \Phi^{1+\alpha} < 0,$$

$$\overline{\mathbf{G}}_{0^+} := \int_{\partial\Omega} \limsup_{s\to 0^+} \frac{sg(s)}{|s|^{1+\alpha}}\, \Phi^{1+\alpha} = \int_{\partial\Omega} \limsup_{s\to 0^+} \sin(s^\beta)\, \Phi^{1+\alpha} = \int_{\partial\Omega} \Phi^{1+\alpha} > 0$$

and then $\underline{\mathbf{G}}_{0^+} < 0 < \overline{\mathbf{G}}_{0^+}$.

Moreover, by looking in (3.1) at the values of $s \in \mathbb{R}$ such that $\lambda(s) = \sigma_1$ we get that (σ_1, u_k) is a solution for any $k \in \mathbb{Z}$, where

$$u_k(x) := \frac{(k\pi)^{1/\beta}}{e+1}(e^x + e^{1-x}),$$

i.e. there is a sequence of solutions of the resonant problem converging to zero, see Fig. 4.

Moreover, computing in (3.1) the local maxima and minima of $\lambda(s)$ we get that (λ_k^*, u_k^*) is a sequence of turning points converging to zero, where

$$\lambda_k^* := \sigma_1 - s_k^{(\alpha-1)/\beta}\, \sin(s_k), \qquad u_k^*(x) := s_k^{1/\beta}(e^x + e^{1-x})$$

and where s_k is such that

$$\tan(s_k) = -\frac{\beta}{\alpha - 1} s_k, \qquad s_k \in [-\pi/2 + k\pi, \pi/2 + k\pi]$$

with $s_k^{1/\beta} \to 0$ as $k \to \infty$ thanks to $\beta < 0$.

Let us observe that the bifurcation from zero phenomena occurs whenever $\alpha > 1$ for any β and that whenever $\alpha + \beta < 1$ the number of oscillations grows up quicker than the number of oscillations of multiples of the eigenfunction and can not be controlled, let us compare Fig. 4 left and right.

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