Quenching phenomena for a non-local diffusion equation with a singular absorption

Raúl Ferreira
Departamento de Matemática Aplicada, Fac. de C.C. Químicas,
U. Complutense de Madrid, 28040 Madrid, Spain
e-mail: raul_ferreira@ucm.es

March 24, 2009

Abstract

In this paper we study the quenching problem for the non-local diffusion equation

$$u_t(x,t) = \int_\Omega J(x-y)u(y,t)\,dy + \int_{\mathbb{R}^N\setminus\Omega} J(x-y)\,dy - u(x,t) - \lambda u^{-p}(x,t).$$

We prove that there exits a critical parameter λ_* such that for all $\lambda > \lambda_*$ every solution quenches and for $\lambda \le \lambda_*$ there are both global and quenching solutions. For the quenching solutions we study the quenching rate and the quenching set. We also prove that the solutions of properly rescaled non local problems approximate the solution of the semilinear heat equation with $u = 1$ at the boundary.

1 Introduction

In this paper we study the quenching phenomena for the problem

$$\begin{cases} u_t = J * u - u - \lambda u^{-p}, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 1, & (x,t) \in \mathbb{R}^N\setminus\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, connected and smooth domain, u_0 is a positive continuous function, $p > 0$ and $*$ denotes convolution. The kernel $J : \mathbb{R}^N \to \mathbb{R}$ is a non-negative, radial, C^1 function with $\int_{\mathbb{R}^N} J(s)\,ds = 1$.

The equation (1.1) is called nonlocal diffusion equation since the diffusion of u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J * u$. As stated in [Fi], if $u(x,t)$ is the density at the point x at time t and $J(x-y)$ is the probability distribution to jump from location y to location x, then

$$J * u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\, dy$$

is the rate at which individuals are arriving to location x and

$$-\int_{\mathbb{R}^N} J(x-y)u(x,t)\, dy = -u(x,t)$$

is the rate at which individuals are leaving location x. If in addition an external source is present, we obtain the evolution equation (1.1). For recent references on non-local diffusion see [BFRW, CC, CER2, CDM, PR] and references therein.

Due to the non-local character of the equation, we need to prescribe the "boundary value" of u not only on the topological boundary $\partial\Omega$, but on the complement of Ω (see [Ch] for Dirichlet condition and [CERW] for Neumann condition).

In section 2 we use a fixed point argument to prove local in time existence and uniqueness of solution of (1.1). Also a comparison principle is established. Let T be the maximal existence time, which may be finite or infinite. If $T < \infty$, then the solution reaches the level $u = 0$ and u_t blows up. This phenomenon is called Quenching. It was studied for the first time in [K] for the problem $v_t = v_{xx} + (1-v)^{-1}$ where quenching happens when v reaches the value $v = 1$. Since then, the phenomenon of quenching for different problems has been the issue of intensive study in recent years, see for example the surveys [C, FL, L1, L2] and the references therein.

Theorem 1.1 *There exists λ_* such that if $\lambda > \lambda_*$ all solutions quench in finite time, whereas for $\lambda \le \lambda_*$ there exists both, global and quenching solutions.*

Once we have characterized for which parameter the solution to problem (1.1) can or cannot quench, we want to study the way the quenching solutions behave as approaching the quenching time. This means that we must investigate the speed at which they quench (the *quenching rate*) and where the solutions quench (the *quenching set*). We define the quenching set as follows

$$Q(u) = \{x \in \Omega;\ \exists\, x_n \to x,\ t_n \nearrow T,\ \text{with } u(x_n,t_n) \to 0\}.$$

We begin with the quenching rate, which is given by the O.D.E. $u_t = -\lambda u^{-p}$. More precisely,

Theorem 1.2 *Let $x_0 \in Q(u)$. Then,*

$$\lim_{t \nearrow T} (T-t)^{\frac{-1}{p+1}} u(x_0,t) = ((p+1)\lambda)^{\frac{1}{p+1}}.$$

To study the quenching set, we first consider the one dimensional symmetric case and we obtain that $Q(u) = \{0\}$ (single point quenching). However, in general it is very difficult to prescribe the quenching set. For instance, given $x_0 \in \Omega$ there exist an initial data such that the quenching set is located in a small ball around x_0.

Theorem 1.3 *i) Let $\Omega = (-L, L)$ and $u_0(x) \in C^1(\mathbb{R})$ be a nonnegative even function such that $u_0' \geq 0$ for $0 \leq x \leq L$. Then $Q(u) = \{0\}$.*

ii) Let $x_0 \in \Omega$. Then, for all $\delta > 0$ there exist a initial data u_0 such that $Q(u) \subset B_\delta(x_0)$.

Finally we compare this type of non-local problem with the local problem,

$$\begin{cases} v_t = \Delta v - \lambda v^{-p}, & (x,t) \in \Omega \times (0, T_v), \\ v(x,t) = 1, & (x,t) \in \partial\Omega \times (0, T_v), \\ v(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

From regularity theory, see [F], we know that if $\Omega \in C^{2+\alpha}$ for some $\alpha \in (0,1)$ and u_0 is smooth enough and satisfies the boundary condition, then $v \in C^{2+\alpha, 1+\alpha/2}(\overline{\Omega} \times [0, T_v))$, where T_v is the maximal time of existence of v.

We take a kernel J which satisfies

$$\int_{\mathbb{R}^N} J(s) s^{2+\alpha} \, ds < \infty. \tag{1.3}$$

Following the ideas of [CER], considering the rescaled kernel,

$$J_\varepsilon(x) = \frac{K_1}{\varepsilon^N} J\left(\frac{x}{\varepsilon}\right), \qquad \text{where} \quad K_1 = \frac{2}{\displaystyle\int_{\mathbb{R}^N} J(s) s^2 \, ds}, \qquad \text{and} \quad \varepsilon > 0,$$

and v_ε the solution of

$$\begin{cases} (v_\varepsilon)_t = \frac{1}{\varepsilon^2}\left(J_\varepsilon * v_\varepsilon - v_\varepsilon\right) - \lambda v_\varepsilon^{-p}, & (x,t) \in \Omega \times (0, T_\varepsilon), \\ v_\varepsilon(x,t) = 1, & (x,t) \in \mathbb{R}^N \backslash \Omega \times (0, T_\varepsilon), \\ v_\varepsilon(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

we have a convergence result that says that for any $\tau > 0$, v_ε converges uniformly to v in sets of the form $\overline{\Omega} \times [0, T_v - \tau]$. Let us observe that we cannot expect that the convergence result extends up to T_v, due to the singularity developed by the absorption term at time $t = T_v$.

Theorem 1.4 *Let J be a kernel which satisfies (1.3), $v \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0, T_v - \tau])$ the solution of* (1.2) *and v_ε the solution of* (1.4)*. Then there exists a positive constant C such that for ε small enough the following estimates holds*

$$\|v - v_\varepsilon\|_{L^\infty(\Omega \times [0, T_v - \tau])} \leq C \varepsilon^\alpha.$$

To end this introduction, we note that the Theorems 1.2 and the first assertion in 1.3 also hold for the local problem (1.2), see for instance [G].

2 Local Existence and uniqueness

Lemma 2.1 *Let $u_0 \in C(\overline{\Omega})$ be a positive function. Then, there exists a unique solution $u \in C^1([0,T) : \Omega)$ of the problem (1.1). Moreover if $T < \infty$ then*

$$\min_{\overline{\Omega}} u(\cdot, T) = 0\,.$$

Proof. This result follows by the Banach fixed-point Theorem. Let X_0 the closed convex subset of the Banach space $C(\overline{\Omega} \times [0, t_0])$ defined by

$$X_0 = \{u \in C(\overline{\Omega} \times [0, t_0]) \,:\, \varepsilon \le u \le K\},$$

where ε and K satisfies $2\varepsilon < u_0(x) < K/2$ and

$$t_0 < \min\{\frac{1}{2}, \frac{\varepsilon}{K + \lambda\varepsilon^{-p}}, \frac{1}{2 + \lambda p \varepsilon^{-p-1}}\}.$$

We introduce the nonlinear operator

$$\begin{aligned} T_{u_0}(u)(x,t) = & \;\; u_0(x) + \int_0^t \left(\int_\Omega J(x-y)u(y,s)\, dy + \int_{\mathbb{R}^N \setminus \Omega} J(x-y)\, dy \right) ds \\ - & \int_0^t (u + \lambda u^{-p})(x,s)\, ds \; = u_0(x) + I_1(x,t) - I_2(x,t) \end{aligned}$$

First, we show that T_{u_0} maps X_0 into X_0. Notice that as $u \ge \varepsilon$ we have that I_2 is continuous. Moreover, as $u \le K$ and $\|J\|_1 = 1$ then I_1 is also continuous. Therefore, $T_{u_0}(u)$ is continuous in $\overline{\Omega} \times [0, t_0]$.

Since I_1 and I_2 are positive, we obtain that, for $t \le t_0$,

$$T_{u_0}(u)(x,t) \ge 2\varepsilon - (K + \lambda\varepsilon^{-p})t_0 \ge \varepsilon$$

and

$$T_{u_0}(u)(x,t) \le K(\frac{1}{2} + t_0) \le K.$$

Now, we prove that T_{u_0} is a strict contraction in X_0.

$$\begin{aligned} \|T_{u_0}(u) - T_{u_0}(v)\|_{X_0} \le & \;\; \left\| \int_0^t \int_\Omega J(x-y)(u(y,s) - v(y,s))\, dy\, ds \right\|_{X_0} \\ & + \left\| \int_0^t (u - v - \lambda(u^{-p} - v^{-p}))(x,s)\, ds \right\|_{X_0} \\ \le & \;\; 2t_0 \|u - v\|_{X_0} + \lambda p \left\| \int_0^t |\xi|^{-p-1} |u - v|(x,s)\, ds \right\|_{X_0} \\ \le & \;\; t_0(2 + \lambda p \varepsilon^{-p-1}) \|u - v\|_{X_0} < \|u - v\|_{X_0}. \end{aligned}$$

Therefore, by Banach's fixed point theorem there exists a unique $w \in X_0$ such that $w = T_{u_0}(w)$. Notice that if we define

$$u = \begin{cases} w, & x \in \Omega, \\ 1, & x \in \mathbb{R}^N \backslash \Omega, \end{cases}$$

we have a solution of (1.1) in the time interval $[0, t_0]$. Now, if $u(\cdot, t_0) > 0$ we can iterate this procedure to extend the solution up to some interval $[t_0, t_1]$. Hence, we conclude that if the maximal existence time is finite, then the solution reaches the level zero at this time. □

Remark 2.1 *Observe that the fixed point w is defined in $\overline{\Omega}$ and, in general, it is different of one at the boundary. Therefore the solution u have a discontinuity on $\partial\Omega$ and the boundary data is not taken in the classical sense, see [ChChR, Ch].*

Remark 2.2 *We also remark that if we consider $u_0 \in C^1(\overline{\Omega})$, we obtain a solution $u \in C^1(\Omega \times [0, T))$. Indeed, following the same argument of previous lemma, we consider*

$$X_0 = \{u \in C([0, t_0] : C^1(\overline{\Omega})) \,:\, \varepsilon \le u \le K \,,\, |u_x| \le M\}$$

and the same operator T_{u_0}. It is easy to see that operator $T_{u_0}(X_0) \subset X_0$ and that it is a strict contraction in X_0. Then, again the Banach fixed-point Theorem gives us the existence and uniqueness of $u \in C^1(\Omega)$.

To end this section we prove a comparison result. To do that we say that $\overline{u}$ is a supersolution if it verifies (1.1) with upper inequalities instead of equalities. More precisely,

Definition 2.1 *A function $\overline{u} \in C(\overline{\Omega} \times [0, T))$ is a supersolution of (1.1) if it a positive function which satisfies*

$$\begin{cases} \overline{u}_t \ge J * \overline{u} - \overline{u} - \lambda \overline{u}^{-p}, & (x, t) \in \Omega \times (0, \overline{T}), \\ \overline{u}(x, t) \ge 1, & (x, t) \in \mathbb{R}^N \backslash \Omega \times (0, \overline{T}), \\ \overline{u}(x, 0) \ge u_0(x), & x \in \Omega. \end{cases} \tag{2.5}$$

Analogously, we say that $\underline{u} \in C(\overline{\Omega} \times [0, \underline{T}))$ is a subsolution if it satisfies (2.5) with the reverse inequalities.

Lemma 2.2 *Let $\overline{u}$ and $\underline{u}$ be a supersolution and a subsolution respectively, then $\overline{u} \ge \underline{u}$ in $\overline{\Omega} \times [0, \underline{T})$.*

Proof. Let δ, M and $t_0 > 0$ three positive parameter such that

$$\min_{\overline{\Omega} \times [0, t_0]} \{\overline{u}, \underline{u}\} = \delta, \qquad \max_{\overline{\Omega} \times [0, t_0]} \{\overline{u}, \underline{u}\} = M$$

Consider the function $z = \overline{u} - \underline{u} + \varepsilon e^{\mu t}$. Notice that $z(x,0) > 0$. Applying the mean value theorem and taking μ large enough, it is easy to see that for $t \in [0, t_0]$

$$\begin{aligned} z_t &\geq \int_{\mathbb{R}^N} J(x-y)z(y,t) - z + \lambda p M^{-p-1} z + \varepsilon e^{\mu t}(\mu - \lambda p \delta^{-p-1}) \\ &> \int_{\mathbb{R}^N} J(x-y)z(y,t) - z + \lambda p M^{-p-1} z. \end{aligned}$$

Now assume that there exists a first time $0 < t_1 < t_0$ such that z vanish at some point $x_1 \in \overline{\Omega}$. At this point we have that $z(\cdot, t_1) \geq 0$ and

$$0 \geq z_t(x_1, t_1) > \int_{\mathbb{R}^N} J(x-y) z(y, t_1)\, dy \geq 0.$$

Therefore, $z > 0$ in $\overline{\Omega} \times [0, t_0]$. Finally, taking the limits $\varepsilon \to 0$ and $\delta \to 0$ we obtain the desired result. □

3 Quenching vs global existence

Lemma 3.1 *If the initial data verifies that*

$$\min u_0(x) \leq \lambda^{1/p}$$

then u quenches in finite time.

Proof. Let $x_0(t)$ be the point such that $\min u(\cdot, t) = u(x_0(t), t)$. Thanks to the restriction on the initial data, at this point we have that u_t is negative. Moreover,

$$u_t(x_0(t), t) \leq 1 - u(x_0(t), t) - \lambda u^{-p}(x_0(t), t) \leq -u(x_0(t), t).$$

Integrating this inequality, we obtain $u(x_0(t), t) \leq u_0(x(0))e^{-t}$. Therefore there exists at time t_0 such that for all $t > t_0$

$$u_t(x_0(t), t) \leq 1 - u(x_0(t), t) - \lambda u^{-p}(x_0(t), t) \leq -\frac{\lambda}{2} u^{-p}(x_0(t), t).$$

Again by integration,

$$u^{p+1}(x_0(t), t) \leq u^{p+1}(x_0(t_0), 0) - \frac{\lambda}{2} t. \tag{3.6}$$

Therefore, the solution quenches at finite time

$$T \leq t_0 + \frac{2}{\lambda} u^{p+1}(x(t_0), t_0).$$

□

Remark 3.1 *Notice that this proof implies that quenching in infinite time is imposible.*

Lemma 3.2 *Let u_1 be the solution of* (1.1) *with $u_0(x) = 1$. If u_1 quenches in finite time, then all solutions of* (1.1) *also quench in finite time.*

Proof. By comparison it is clear that if $\|u_0\|_\infty \le 1$ then u quenches in finite time. In other case, we compare with the function

$$U(t) = (\lambda p)^{\frac{1}{1+p}}(A - t)^{\frac{1}{1+p}} ,$$

which is a solution of (1.1) in Ω. Now, taking A large, we get that $U(0) \ge \|u_0\|_\infty$. Then, by comparison we have that $U \ge u$ as long as $U \ge 1$. So, there exist a time t_0 such that $1 = U(t_0) \ge u(\cdot, t_0)$. □

Lemma 3.3 *Let u_1 be the solution of* (1.1) *with $u_0(x) = 1$. It satisfies that, either it quenches in finite time, either it converges to a stationary solution.*

Proof. First we observe that $v = (u_1)_t$ satisfies

$$\begin{cases} v_t = J * v - v + p\lambda u^{-p-1} v, & (x,t) \in \Omega \times (0,T), \\ v(x,t) = 0, & (x,t) \in \mathbb{R}^N \backslash \Omega \times (0,T), \\ v(x,0) = -\lambda, & x \in \Omega. \end{cases} \tag{3.7}$$

Then, by comparison we obtain that $v = u_t \le 0$.

Now, by lemma 3.1, if we suppose that u is a global solution, then we have that $\lambda^{1/p} \le u \le 1$. Therefore, $u \to u_\infty$ as $t \to \infty$. Moreover, we have the Lyapunov functional

$$\mathcal{F}[u](t) = \frac{1}{4} \int\int J(x-y)(u(x,t) - u(y,t))^2\, dxdy + \frac{\lambda}{1-p} \int u^{1-p}(x,t)\, dx,$$

which satisfies

$$\frac{d}{dt}\mathcal{F}[u](t) = -\int (u_t)^2(x,t)\, dx,$$

see [IR]. Then, u_∞ must be a stationary solution. □

Now we study the stationary solutions. Let us denote w_λ the solution of the problem

$$\begin{cases} J * w - w - \lambda w^{-p} = 0, & x \in \Omega, \\ w = 1, & x \in \mathbb{R}^N \setminus \Omega \end{cases} \tag{3.8}$$

Lemma 3.4 *If $\lambda \le \lambda_*$ there exists at least one solution of* (3.8) *, while for $\lambda > \lambda_*$ no stationary solution exists.*

Proof. The proof is given in several steps.

1. From lemma 3.1 and the proof of lemma 3.3 we have the following a-priori estimates

$$\lambda^{1/p} \le w_\lambda(x) \le 1\,.$$

2. For $\lambda > 1$ no stationary solution exists.

Let u_1 be the solution with $u_0(x) = 1$. We have that

$$(u_1)_t = J * u_1 - u_1 - \lambda u_1^{-p} \le 1 - \lambda.$$

Then, it quenches in finite time. So, by lemma 3.3 no stationary solution exists.

3. If λ small there exists a stationary solution.

Observe that for $\lambda = 0$, $w_0 \equiv 1$ is a solution. Now, we linearize around this solution. Let $A = \{v \in C(\overline{\Omega}) \,:\, -\varepsilon < v < 1\} \subset C(\overline{\Omega})$ and $F(\lambda, v) : (-\varepsilon, \varepsilon) \times A \longrightarrow C(\overline{\Omega})$ given by

$$F(\lambda, v)(x,t) = \int_\Omega J(x-y)v(y,t)\,dy - v(x,t) + \lambda(1 - v(x,t))^{-p}.$$

Observe that this is a differentiable function and $F(0,0) = 0$. Moreover,

$$F_v(0,0)(z)(x,t) = \int_\Omega J(x-y)z(y,t)\,dy - z(x,t),$$

is a continuous linear operator. As its kernel is the function $z = 0$, then it is injective. On the other hand,

$$L(z)(x,t) = \int_\Omega J(x-y)z(y,t)\,dy$$

is a compact linear operator. Hence $F_v(0,0)$ is bijective. Finally, by the Open-Mapping Theorem we deduce that $F_v(0,0)$ is a homeomorphism of $C(\overline{\Omega})$ into $C(\overline{\Omega})$. Therefore, we can apply the Implicit Function Theorem to ensure that for λ small enough there exists a solution $v_\lambda \in A$ which is close to $v_0 = 0$, see [CR]. Then, $w_\lambda = 1 - v_\lambda$ is a continuous solution of (3.8).

4. Let λ_1 such that w_{λ_1} exists, then w_λ exists for all $\lambda < \lambda_1$. Moreover, $w_\lambda \ge w_{\lambda_1}$.

This follows from the fact that w_{λ_1} is a subsolution of problem (1.1). Then, u_1 is bounded from bellow. Therefore, by lemma 3.3 it converges to a stationary solution which is bigger than w_{λ_1}.

5. Let $\lambda_* = \sup\{\lambda : w_\lambda \text{ exists}\}$. Thanks to the monotonicity property given in the last step, it is easy to check that

$$w_{\lambda_*} = \lim_{\lambda \nearrow \lambda_*} w_\lambda$$

is a solution of (3.8). □

4 Quenching rates and quenching set

In this section we study the behaviour of u near the quenching time. We begin with the quenching rate.

Proof of Theorem 1.2.

To obtain the upper estimates we observe that

$$u^p u_t = u^p\Big(J * u - u - \lambda u^{-p}\Big) \geq -u^{p+1} - \lambda \geq -1 - \lambda.$$

Taking $x_0 \in Q(u)$ and integrating the above inequality between t and T we obtain

$$u^{p+1}(x_0, t) \leq C(T - t)\,. \tag{4.9}$$

Using this inequality, we have that

$$u^p u_t(x_0, t) \geq -u^{p+1}(x_0, t) - \lambda \geq -C(T - t) - \lambda.$$

Again, by integration

$$u^{p+1}(x_0, t) \leq (p + 1)\lambda(T - t)\Big(1 + C(T - t)\Big).$$

In order to obtain the lower estimate, we note that

$$u^p u_t = u^p\Big(J * u - u - \lambda u^{-p}\Big) \leq u^p - \lambda.$$

Using (4.9) we obtain that

$$u^p u_t(x_0, t) \leq C(T - t)^{\frac{p}{p+1}} - \lambda.$$

Integrating this inequality,

$$u^{p+1}(x_0, t) \geq (p + 1)\lambda(T - t)\Big(1 - C(T - t)^{\frac{p}{p+1}}\Big).$$

Summing up, we have

$$(p + 1)\lambda\Big(1 - C(T - t)^{\frac{p}{p+1}}\Big) \leq \frac{u^{p+1}(x_0, t)}{(T - t)} \leq (p + 1)\lambda\Big(1 + C(T - t)\Big).$$

□

Next, we study the quenching set. First, we prove the following estimate.

Lemma 4.1 *Let x_1 and x_2 be two quenching points and assume that*

$$\int_{\mathbb{R}^N} J(x_1 - y)u(y,t)\,dy > \int_{\mathbb{R}^N} J(x_2 - y)u(y,t)\,dy \qquad t \in [t_0, T).$$

Then $u(x_1,t) < u(x_2,t)$ for $t_0 \le t < T$.

Proof. We define the function $w(t) = u(x_1,t) - u(x_2,t)$ and assume that $w(t_0) \ge 0$. Since w satisfies

$$w' = \int_{\mathbb{R}^N} \Big(J(x_1 - y) - J(x_2 - y) \Big) u(y,t)\,dy - w - \lambda(u^{-p}(x_1,t) - u^{-p}(x_2,t)),$$

we observe that if $w(t_1) = 0$ for some $t_1 \in [t_0, T)$, then $w'(t_1) > 0$. Hence, $w > 0$ in (t_0, T). Using this fact, we obtain that

$$w' > -w.$$

Finally, integrating this inequality between $(t_0 + T)/2$ and T, we obtain that

$$w(T) > w\left(\frac{t_0 + T}{2}\right) e^{-(T - t_0)/2} > 0,$$

which is a contradiction with that fact that $w \to 0$ as $t \to T$. □

In order to prove Theorem 1.3, we start with the one dimensional symmetric case

Lemma 4.2 *Let $\Omega = (-L, L)$ and let $u_0(x) \in C^1(\mathbb{R})$ be a nonnegative even function such that $u_0'(x) \ge 0$ for $0 \le x \le L$. Then $Q(u) = \{0\}$.*

Proof. First we note that as J is a symmetric function we have that $u(\cdot, t)$ is symmetric for all time. Now we prove that u is an increasing function in $[0, L]$. To do that we observe that $v = u_x$ satisfies

$$\begin{cases} v_t = \displaystyle\int_{\mathbb{R}} J'(x - y)u(y,t)\,dy - v + \lambda u^{-p-1} v & x \in (-L, L), \\ v = 0 & x \in \mathbb{R} \setminus (-L, L). \end{cases}$$

Since u is symmetric and J' is odd, we have that

$$\begin{aligned} \int_{\mathbb{R}} J'(x - y)u(y,t)\,dy &= \int_x^{\infty} J'(x - y)u(-y,t)\,dy - \int_{-\infty}^{x} J'(y - x)u(y,t)\,dy \\ &= \int_{-\infty}^{0} J'(z)\Big(u(z - x, t) - u(z + x, t)\Big)\,dz. \end{aligned}$$

Using again the symmetry of u, we have that this integral is strictly negative for $-L < x < 0$, zero at $x = 0$ and strictly positive for $0 < x < L$. Therefore, if we start with

$v_0(x) \leq 0$ for $-L < x < 0$, $v_0(0) = 0$ and $v_0(x) \geq 0$ in $0 < x < L$, then it holds that $v(x,t) < 0$ for $-L < x < 0$, $v(0,t) = 0$ and $v(x,t) > 0$ for $0 < x < L$.

In order to prove that the only quenching point is the origin, we observe that for all $x \in \overline{\Omega}$

$$\int_{\mathbb{R}} J(-y)u(y,t)\,dy < \int_{\mathbb{R}} J(x-y)u(y,t)\,dy \qquad \text{and} \qquad \min_{\overline{\Omega}} u(\cdot,t) = u(0,t).$$

Then, from the previous Lemma we obtain that $Q(u) = \{0\}$. □

Lemma 4.3 *Let $x_0 \in \Omega$. Then, for all $\delta > 0$ there exist a initial data u_0 such that the solution of* (1.1) *quenches and $Q(u) \subset B_\delta(x_0)$.*

Proof. We take $x_0 \in \Omega$ and we consider an initial data such that

$$\min u_0(x) = u_0(x_0) < \left(\frac{\lambda}{2}\right)^{1/p}$$

By lemma 3.1 the solution quenches in finite time. Moreover, in the proof of this lemma we can take $t_0 = 0$ and from (3.6) we have the following estimation of the quenching time

$$T \leq \frac{2}{\lambda} u_0^{p+1}(x_0).$$

On the other hand,

$$u^p u_t = u^p(J * u - u - \lambda u^{-p}) \geq -(1+\lambda).$$

Integrating this inequality between $t = 0$ and $t = T$,

$$u^{p+1}(x,T) \geq u_0^{p+1}(x) - (1+\lambda)T$$

Therefore,

$$u_0^{p+1}(x) > \frac{2(1+\lambda)}{\lambda} u_0^{p+1}(x_0) \geq (1+\lambda)T,$$

which implies that $x \notin Q(u)$.

Then, taking $u_0(x)$ such that in the complement of $B_{x_0}(\delta)$ satisfies

$$u_0(x) > \left(\frac{2(1+\lambda)}{\lambda}\right)^{1/(p+1)} u_0(x_0),$$

we obtain the desired result. □

5 Non-local vs local diffusion

In this section we compare the solution v of the local problem (1.2) with the solution v_ε of the nonlocal problem (1.4). To do that, we need to define v in the complement of Ω. Let $\tilde{v}$ be a $C^{2+\alpha,1+\alpha/2}$ extension of v to $\mathbb{R}^N \times [0,T)$, such that

$$\tilde{v}(x,t) = \begin{cases} v(x,t), & (x,t) \in \Omega \times [0,T), \\ 1 + O(\varepsilon), & (x,t) \in \mathbb{R}^N \backslash \Omega \times [0,T). \end{cases}$$

Taking into account that the kernel J satisfies (1.3), we can adapt the proof given in [CER]. We obtain that for every function $\tilde{v} \in C^{2+\alpha,1+\alpha/2}$, we have

$$\max_{t\in[0,t_0]} \|\Delta \tilde{v} - \frac{1}{\varepsilon^2}(J_\varepsilon * \tilde{v} - \tilde{v})\|_{L^\infty(\Omega)} = O(\varepsilon^\alpha), \tag{5.10}$$

for all $0 < t_0 < T_v$. This gives us that if we take the non-local operator as an approximation to the laplacian, then it is consistent.

This consistence allows us to prove the convergence result.

Proof of Theorem 1.4. We define the error function as $e(x,t) = v(x,t) - v_\epsilon(x,t)$.

Let c be a constant such that $v \geq c$ for every $t \in [0, T - \tau]$ and

$$\widetilde{t} = \max\{t \in [0, T-\tau] \text{ such that } \|e\|_{L^\infty(\Omega)}(t) \leq c/2\} \tag{5.11}$$

so as to ensure that, up to time $\widetilde{t}$, none of the solutions, neither v nor v_ε, quench.

From (5.10) we have that the error function satisfies

$$\begin{cases} e_t = \frac{1}{\varepsilon^2}(J_\varepsilon * e - e) - \lambda(\tilde{v}^{-p} - v_\varepsilon^{-p}) + O(\varepsilon^\alpha), & (x,t) \in \Omega \times (0,\widetilde{t}), \\ e = O(\varepsilon), & (x,t) \in \mathbb{R}^N \setminus \Omega \times (0,\widetilde{t}). \end{cases}$$

Applying the Mean Value Theorem to the nonlinear term, we have that for ξ between $\tilde{v}$ and v_ε,

$$\begin{aligned} e_t = & \ \tfrac{1}{\varepsilon^2}(J_\varepsilon * e - e) + \lambda p \xi^{-p-1} e + O(\varepsilon^\alpha) \\ \leq & \ \tfrac{1}{\varepsilon^2}(J_\varepsilon * e - e) + \lambda p \left(\tfrac{2}{c}\right)^{p+1} e + O(\varepsilon^\alpha). \end{aligned}$$

Therefore, e is a subsolution of following problem

$$\begin{cases} w_t = \frac{1}{\varepsilon^2}(J_\varepsilon * w - w) + C_1 w + C_2 \varepsilon^\alpha, & (x,t) \in \Omega \times (0,\widetilde{t}), \\ w(x,t) = C_3 \varepsilon, & (x,t) \in \mathbb{R}^N \setminus \Omega \times (0,\widetilde{t}), \\ w(x,0) = 0, & x \in \Omega. \end{cases} \tag{5.12}$$

Now, we consider the function

$$\overline{w}(t) = \varepsilon^\alpha \frac{2C_2}{C_1}\left(e^{C_1 t} - \frac{1}{2}\right),$$

which satisfies (5.12) for $x \in \Omega$, while for $x \notin \Omega$ we have that for ε small enough

$$\overline{w}(t) \geq \overline{w}(0) = \varepsilon^{\alpha} \frac{C_2}{C_1} \geq C_3 \varepsilon \, .$$

Thus, $\overline{w}$ is a supersolution of (5.12) and by comparison $e \leq \overline{w}$.

Arguing in same way with $-e$ we arrive at

$$\|e\|_{L^{\infty}(\Omega \times [0,\widetilde{t}])} \leq \varepsilon^{\alpha} \frac{2C_2}{C_1} \left(e^{C_1 t} - \frac{1}{2} \right) ,$$

from which it is immediate to see that $\widetilde{t} = T_v - \tau$. □

Acknowledgements

Supported by grant CCG07-UCM/ESP-2393 UCM-Comunidad de Madrid and MTM2008-06326-C02-02, MCI Spain.

References

[BFRW] Bates, P., Fife, P., Ren, X. and Wang, X. *Travelling waves in a convolution model for phase transitions.* Arch. Ration. Mech. Anal. **138** (1997), 105-136.

[CC] Carr, J. and Chmaj, A. *Uniqueness of travelling waves for nonlocal monostable equations.* Proc. Amer. Math. Soc. **132** (2004), 2433-2439.

[CER] Cortazar, C., Elgueta, M. and Rossi, J. D. *Nonlocal diffusion Problems that approximate the heat equation with Dirichlet boundary conditions*, To appear in Israel J. Math.

[CER2] Cortazar, C., Elgueta, M. Rossi, J. D. *A nonlocal diffusion equation whose solutions develop a free boundary.* Ann. Henri Poincaré **6** (2005), 269–281.

[CERW] Cortazar, C., Elgueta, M., Rossi, J. D. and Wolanski, N. *Boundary fluxed for non-local diffusion.* J. Differential Equations **234** (2007), 360–390

[CR] Crandall, M. G. and Rabinowitz, P. H. *Bifurcation from simple eigenvalues.* J. Functional Analysis, **8** (1971), 321–340.

[C] Chan, C. Y. *Recent advances in quenching phenomena.* Proc. Dynam. Systems. Appl. **2** (1996), 107–113.

[ChChR] Chasseigne, E., Chaves, M. and Rossi, J. D. *Asymtotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl. **86** (2006), 271–291.

[Ch] Chasseigne, E. *The Dirichlet problem for some nonlocal diffusion equations.* Differential Integral Equations **20** (2007), 1389–1404.

[CDM] Coville, J., Dávila, J. and Martínez, S. *Existence and uniqueness of solutions to a nonlocal equation with monostable nonlinearity.* SIAM J. Math. Anal. **39** (2008), 1693–1709.

[FL] Fila, M. and Levine, H. A. *Quenching on the boundary.* Nonlinear Anal. **21** (1993), 795–802.

[F] Friedman, A. *Partial Differential Equation of Parabolic Type.* Prentice-Hall, Englewood Cliffs, 1969.

[Fi] Fife, P. *Some nonclassical trends in parabolic and parabolic-like evolutions.* Trends in nonlinear analysis, 153–191, Springer, Berlin, 2003.

[G] Guo, J. S. *On the quenching rate estimate.* Quart. Appl. Math. **49** (1991), 747–752.

[IR] Ignat, L. I. and RossiJ. D. *Decay estimates for nonlocal problems via energy methods*, preprint.

[K] Kawarada, H. *On solutions of initial-boundary problem for* $u_t = u_{xx} + 1/(1-u)$. Publ. Res. Inst. Math. Sci. **10** (1974/75), 729–736.

[L1] Levine, H. A. *The phenomenon of quenching: a survey.* In "Trends in the Theory and Practice of Nonlinear Analysis", (V. Lakshmikantham, ed.), Elsevier Science Publ., North Holland, 1985, pp. 275–286.

[L2] Levine, H. A. *Quenching and beyond: a survey of recent results.* GAKUTO Internat. Ser. Math. Sci. Appl. **2** (1993) , Nonlinear mathematical problems in industry II , Gakkōtosho, Tokyo, 501–512.

[PR] Pérez-Llanos, M. and Rossi, J. D. *Blow-up for a non-local diffusion problem with Neumann boundary conditions and a reaction term.* To appear in Nonlinear Analysis TM&A