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Perturbation of analytic semigroups in scales of Banach spaces and applications to parabolic equations with low regularity data *

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1 Introduction

When dealing with partial differential equations, either linear or nonlinear, and mostly for equations of elliptic or parabolic type, it is often found that the equation can be solved when the "data" of the problem are not in a single, but in several functions spaces among which we can chose. In other words, in a natural fashion, we face equations that can be set in "scales" of spaces. For example, when solving the linear heat equation with, say, Dirichlet boundary conditions in a bounded smooth domain Ω, the initial data can be taken in $L^q(\Omega)$, for $1 \leq q \leq \infty$. The solution remains in the same space and enters in fact in some other $L^p(\Omega)$ space with $p > q$ at a precise rate, which are the classical estimates for the heat equation, see e.g. [8].

Restricting to parabolic equations, in his seminal monograph [11], D.Henry, made systematic the use of the scale of fractional power spaces associated to the elliptic part of the equation; see also [10] for early applications of these spaces. This abstract approach has proven very rich and flexible in dealing with enormous classes of parabolic problems. Moreover it provided tools for a real geometric theory for the dynamics of parabolic equations.

Closer to more concrete parabolic problems with elliptic part in divergence form, H. Amann, using techniques of interpolation and extrapolation spaces worked out a rather

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complete framework for these problems with smooth coefficients, using the Bessel potential spaces; see [2] for a complete survey.

Similar approach to the ones above can be found in [12] as well. Also, abstract results in this direction are compiled in [3]; see in particular Chapter 5 in the latter reference.

Although it is very often disregarded, the fact that the same problem can be set in a family of spaces in a scale, is a valuable property since, among many other questions, allows to choose the right space of the scale to "fit" to a particular choice of the data. Another outcome of this approach is, for example, that one finds more precise information on the smoothing effect of the equation, since one can perform in a natural way bootsrapping argument on the scale.

In this context, in this paper we concentrate in some perturbation results for linear parabolic problems in certain scales of spaces. Hence, at an abstract level, we assume we have given a linear semigroup on a scale of spaces and seek to add suitable linear perturbations in such a way that we still obtain a well defined class of solutions of the perturbed problem, defining a perturbed linear semigroup on the same original scale of spaces. The parabolic nature of the original problem (and of the perturbed one) is reflected in the fact that the original semigroup takes elements of one space in the scale into other spaces in the scale; see Section 3 and more precisely (3.3). The perturbations we consider are linear and continuous transformation between two spaces of the scale, see (3.8), and the rule of thumb is that a given perturbation would determine the spaces of initial data in the scale for which the perturbed problem can be solved, as well as the spaces in the scale into which the solution smoothes, see Theorem 3.13 and Proposition 3.15. We also analyze the question of robustness of the estimates and the continuity of the resulting semigroup with respect of the perturbation, see Theorem 4.1. Observe that our approach to these perturbations results is based on the variation of constants formula, (3.9), and the smoothing properties of the original semigroup rather than to "elliptic" properties of the infinitesimal generator of the semigroup. In fact as it will be shown below the "parabolic" approach gives sharper results than the "elliptic" one; compare Theorems 3.13 and 3.20. Note that the basic assumption (3.3) is satisfied for the fractional power spaces associated to any sectorial operator as in [11]. Hence all the results in Sections 3 and 4 apply in this setting as well. Note also that we also obtain results on the linear nonhomogenous problems in Theorem 3.7 and in Remark 3.17.

These abstract results are motivated, with no doubt, for a systematic study and applications of partial differential equations of parabolic type and more precisely in considering low regularity perturbations in the coefficients and/or the boundary conditions of the problems.

Hence, in Section 2 we first review and collect some of the results in [2] for parabolic problems with smooth coefficients. Note that in this reference, regularity of the coefficients is used in an essential way. The same happens in [12].

Then, the abstract results in Sections 3 and 4 are applied in Section 5 to the problems in Section 2 and also to some parabolic problems in unbounded domains in Section 6. In the latter case we discuss several settings including Lebesgue or Bessel scales and even uniform spaces. Linear nonhomogeneous problems, using Theorem 3.7, are discussed in

Remarks 5.8, 6.3, 6.6, 6.11 and 6.16. Also, in Section 7 we show how to obtain results for the underlying elliptic problems.

Finally in Section 8 we discuss how to apply the results in this paper to semigroups that are not strongly continuous, that is, not continuous at $t = 0$ or to nonanalytic semigroups. Also, we consider the case of semigroups with "defect", that is, singular semigroups at $t = 0$.

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2 Parabolic problems with smooth coefficients

Let Ω be an open bounded smooth set in $\mathbb{R}^N$ with a C^2 boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be a smooth subset of the boundary, isolated from the rest of the boundary, that is, $\mathrm{dist}(\Gamma, \partial\Omega \setminus \Gamma) > 0$.

Consider the problem

$$\left\{ \begin{array}{rcll} u_t - \mathrm{div}(a(x)\nabla u) + c(x)u &=& 0 & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& 0 & \text{on } \Gamma \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \setminus \Gamma \\ u(0) &=& u_0 & \text{in } \Omega \end{array} \right. \tag{2.1}$$

where $a \in C^1(\overline{\Omega})$ with $a(x) \geq a_0 > 0$ in Ω, $c \in C^1(\overline{\Omega})$ and $\mathcal{B}$ denotes the boundary operator in $\partial\Omega \setminus \Gamma$

$$\mathcal{B}u = u, \quad \text{Dirichlet case,} \quad \text{or} \quad \mathcal{B}u = a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u, \quad \text{Robin case,}$$

being $\vec{n}$ the outward normal vector-field to $\partial\Omega \setminus \Gamma$ and $b(x)$ a $C^1(\partial\Omega)$ function.

For this, denote by A_0 the operator $A_0 u = -\mathrm{div}(a(x)\nabla u) + c(x)u$ with boundary conditions $a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u = 0$ on Γ and $\mathcal{B}u = 0$ on $\partial\Omega \setminus \Gamma$. Note the coefficients a, b, c are C^1–smooth. Also, note that all the analysis below applies in the case the diffusion coefficient is a positive definite matrix instead of a scalar coefficient. We deal with the latter case here only because the notations become simpler.

Choosing $L^q(\Omega)$, for $1 < q < \infty$, as a base space, the unbounded linear operator $-A_0 : D(A_0) \subset L^q(\Omega) \to L^q(\Omega)$, with domain $D(A_0) = H^{2,q}_{bc}(\Omega)$, consisting of all functions in $H^{2,q}(\Omega)$ which satisfy all boundary conditions above, generates an analytic semigroup in $L^q(\Omega)$, see [2]. Here and below $H^{s,q}(\Omega)$ denote the Bessel potentials spaces which coincide with the usual Sobolev spaces for integer s if $1 < q < \infty$ or for all s if $q = 2$.

Using the complex interpolation–extrapolation procedure, one can construct the scale of Banach spaces associated to this operator, which will be denoted $H^{2\alpha,q}_{bc}(\Omega)$ for $\alpha \in [-1, 1]$, which are closed subspaces of $H^{2\alpha,q}(\Omega)$ incorporating some of the boundary conditions. In particular, we have $H^{0,q}_{bc}(\Omega) = L^q(\Omega)$, and

$$H^{1,q}_{bc}(\Omega) = \left\{ \begin{array}{ll} \{u \in H^{1,q}(\Omega) : \ u = 0 \text{ in } \partial\Omega \setminus \Gamma\} & \text{for } \mathcal{B} \text{ Dirichlet} \\ H^{1,q}(\Omega) & \text{for } \mathcal{B} \text{ Robin.} \end{array} \right.$$

Recall that Bessel spaces have the sharp embeddings

$$H^{s,q}(\Omega) \subset \begin{cases} L^r(\Omega), \ s - \frac{N}{q} \geq -\frac{N}{r}, \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Omega), \ \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\bar{\Omega}) & \text{if } s - \frac{N}{q} > \eta > 0 \end{cases}$$

with continuous embeddings, see [1]. This embeddings are known to be optimal.

Also, if γ_Γ denotes the trace operator on Γ, then for $s > \frac{1}{q}$, γ_Γ is well defined on $H^{s,q}(\Omega)$ and

$$H^{s,q}(\Omega) \overset{\gamma_\Gamma}{\to} \begin{cases} L^r(\Gamma), \ s - \frac{N}{q} \geq -\frac{N-1}{r}, \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ L^r(\Gamma), \ \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta(\Gamma) & \text{if } s - \frac{N}{q} > \eta > 0 \end{cases}$$

see [1].

Note that the scale with negative exponents satisfies $H^{-2\alpha,q}_{bc}(\Omega) = (H^{2\alpha,q'}_{bc}(\Omega))'$, for $0 < \alpha < 1$. Moreover, we have $H^{-2\alpha,q}(\Omega) = (H^{2\alpha,q'}(\Omega))'$ and $H^{-2\alpha,q}(\Omega) \hookrightarrow H^{-2\alpha,q}_{bc}(\Omega)$. See [2] for details.

Using this it is easy to obtain that for $s > 0$ we have

$$H^{-s,q}(\Omega) \supset \begin{cases} L^r(\Omega), \ -s - \frac{N}{q} \leq -\frac{N}{r}, \ 1 < r \leq \infty, & \text{if } -s - \frac{N}{q} > -N \\ L^r(\Omega), \ \ 1 < r \leq \infty, & \text{if } -s - \frac{N}{q} = -N \\ \mathcal{M}(\Omega) & \text{if } -s - \frac{N}{q} < -N. \end{cases}$$

Then, the operator $-A_0$ or, more precisely, a suitable realization of it, generates an analytic semigroup, $S_0(t)$, in each space of the scale $H^{2\alpha,q}_{bc}(\Omega)$, $\alpha \in [-1, 1]$. This semigroup is order preserving and satisfies the smoothing estimates

$$\|S_0(t)u_0\|_{H^{2\alpha,q}_{bc}(\Omega)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\beta,q}_{bc}(\Omega) \tag{2.2}$$

for $1 \geq \alpha \geq \beta \geq -1$ and some $\mu \in I\!R$. In particular, one has

$$\|S_0(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\tau,\rho}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \qquad t > 0, \quad u_0 \in L^\rho(\Omega) \tag{2.3}$$

for $1 \leq \rho \leq \tau \leq \infty$. For any u_0 in $H^{2\beta,q}_{bc}(\Omega)$ or $L^\rho(\Omega)$, the function $u(t; u_0) := S_0(t)u_0$, $t > 0$, is a classical solution of (2.1). The reader is referred to [2] and references therein, for further properties of these spaces and semigroups.

Note that this construction applies to much more general elliptic operators than above. Also, in the construction above the regularity of the coefficients, plays a fundamental role; see [2].

3 Perturbation of linear analytic semigroups in scales of Banach spaces

From the example of Section 2, note that (2.2), can be rewritten in an abstract language as follows. For this we denote

$$X^\alpha := H^{2\alpha,q}_{bc}(\Omega), \quad \alpha \in I := [-1,1] \tag{3.1}$$

and write (2.2) as

$$\|S_0(t)\|_{\mathcal{L}(X^\beta,X^\alpha)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}, \quad \alpha \geq \beta.$$

Analogously, (2.3) can be written in the same way for the scale of Lebesgue spaces $X^\alpha = L^q(\Omega)$ with $q = -\frac{N}{2q}$ with $\alpha \in I := [-N/2, 0]$.

Hence, in this section we consider a linear analytic semigroup $S(t)$ defined on each of the spaces of the family of Banach spaces (the "scale") $\{X^\alpha\}_{\alpha\in I}$ where I is an interval of real indexes. The norm of the space X^α is denoted by $\|\cdot\|_\alpha$.

Note that no other functional relationship is assumed among the spaces of the scale, unless otherwise stated. Sometimes we will assume that the spaces are "nested", that is, for all $\alpha, \beta \in I$ with $\alpha \geq \beta$ we have

$$X^\alpha \subset X^\beta \tag{3.2}$$

with continuous inclusion and the norm of the inclusion will be denoted $\|i\|_{\alpha,\beta}$. In such a case we will say, for short, that the scale is nested. This situation will be explicitly cited when needed. Note that for the example above, (3.1), we have $\|i\|_{\alpha,\beta} \leq 1$ for all α, β.

We also assume the semigroup acting on the scale satisfies, $\alpha, \beta \in I$ with $\alpha \geq \beta$

$$\|S(t)\|_{\beta,\alpha} := \|S(t)\|_{\mathcal{L}(X^\beta,X^\alpha)} \leq \frac{M_0(\beta,\alpha)}{t^{\alpha-\beta}}, \qquad \text{for all} \quad 0 < t \leq 1 \tag{3.3}$$

for some constant $M_0(\beta, \alpha) > 0$.

Remark 3.1
i) Note that the semigroup $S_0(t)$ of Section 2 in the scale (3.1) satisfies that for each β, the domain of the generator $-A_0$ in X^β is given by $D(A_0) = X^{\beta+1}$ and also the inclusion (3.2) is dense and compact. These properties will not be used below; see however Theorem 3.20.
ii) The analysis we carry out below is based on (3.3). Note that this condition can be relaxed assuming that for any given $\beta \in I$ there exist a set $R_0(\beta) \subset I$ such that for any $\alpha \in R_0(\beta)$ we have $\alpha \geq \beta$ and (3.3). Note that $\beta \in R_0(\beta)$ and $R_0(\beta)$ stands for the set of spaces for which the evolution operator smoothes, starting from X^β. Also note that it is not essential that I is an interval. In fact (3.3) corresponds to the case when $R_0(\beta) = [\beta, \infty) \cap I$ for all $\beta \in I$.

Most of the results below can be easily adapted to this more general situation.

Observe that from these assumptions we get

Lemma 3.2 *Assume (3.3) is satisfied. Then*
i) For every $\alpha, \beta \in I$ *and* $\alpha \geq \beta$ *and for all* $T > 0$,

$$\|S(t)\|_{\beta,\alpha} \leq \frac{M_0(\beta,\alpha,T)}{t^{\alpha-\beta}}, \qquad \textit{for all} \quad 0 < t \leq T \tag{3.4}$$

for some constant $M_0(\beta,\alpha,T) > 0$.
ii) For each $\beta \in I$ *there exists* $\omega(\beta) \geq 0$ *such that*

$$\|S(t)\|_{\beta,\beta} \leq M_0(\beta,\beta)\mathrm{e}^{\omega(\beta)t}, \quad \textit{for all} \quad t > 0$$

and for every $\alpha, \beta \in I$ *and* $\alpha \geq \beta$ *there exists* $\omega = \omega(\beta,\alpha)$ *and* $M(\beta,\alpha)$ *such that*

$$\|S(t)\|_{\beta,\alpha} \leq \frac{M(\beta,\alpha)\mathrm{e}^{\omega t}}{t^{\alpha-\beta}}, \qquad \textit{for all} \quad 0 < t < \infty.$$

iii) Assume the scale is nested, that is (3.2). Then, if for some fixed $\beta_0 \in I$, *we have*

$$\|S(t)\|_{\beta_0,\beta_0} \leq M\mathrm{e}^{\omega_0 t}, \quad \textit{for all} \quad t > 0 \tag{3.5}$$

for some $M = M(\beta_0)$ *and* $\omega_0 \in I\!R$, *then for any* $\alpha \in I$, *there exists a constant* $M(\alpha) \geq 1$ *such that*

$$\|S(t)\|_{\alpha,\alpha} \leq M(\alpha)\mathrm{e}^{\omega_0 t}, \quad \textit{for all} \quad t > 0. \tag{3.6}$$

Moreover, given $t_0 > 0$, *define* $\delta = \|S(t_0)\|_{\beta_0,\beta_0}$. *Then we have (3.5) with*

$$\omega_0 = \frac{\ln(\delta)}{t_0}$$

and some constant M *depending on* t_0, δ *and* $M_0(\beta_0,\beta_0,t_0)$ *as in (3.4). In particular if* $\delta < 1$ *then* $\omega_0 < 0$.
iv) In the situation of iii), for every $\alpha, \beta \in I$ *and* $\alpha \geq \beta$ *we have*

$$\|S(t)\|_{\beta,\alpha} \leq \begin{cases} M_1(\beta,\alpha)t^{-(\alpha-\beta)} & \textit{if} \ \ 0 < t \leq 1, \\ M_1(\beta,\alpha)\mathrm{e}^{\omega_0 t} & \textit{if} \ \ t > 1 \end{cases}$$

for some positive constant $M_1(\beta,\alpha)$.

In particular, for all $\varepsilon > 0$ *there exists* $M_\varepsilon(\beta,\alpha) > 0$ *such that*

$$\|S(t)\|_{\beta,\alpha} \leq M_\varepsilon(\beta,\alpha)\frac{\mathrm{e}^{(\omega_0+\varepsilon)t}}{t^{\alpha-\beta}}, \quad \textit{for all} \quad t > 0.$$

Proof.
i) Indeed, given $T > 0$ define n as the smallest integer such that $T \leq n+1$. Then, for $0 < t \leq T$, define $h = \frac{t}{n+1} \leq 1$ and $s_j = jh$, $j = 0, \ldots, n+1$. Thus $s_{n+1} = t$ and, since

$$S(t) = S(s_{n+1} - s_n) \cdots S(s_1 - s)$$

we get, from (3.3),

$$\|S(t)\|_{\beta,\alpha} \leq M_0(\alpha,\alpha)^n M_0(\beta,\alpha)(n+1)^{\alpha-\beta} t^{-(\alpha-\beta)} \qquad \text{for all} \quad 0 < t \leq T.$$

Hence we can take

$$M_0(\beta,\alpha,T) = M_0(\alpha,\alpha)^n M_0(\beta,\alpha)(n+1)^{\alpha-\beta}.$$

ii) In particular, with $\alpha = \beta$, given $t > 0$ define $n \in I\!N$ such that $n \leq t < n+1$ and we get as above,

$$\|S(t)\|_{\beta,\beta} \leq M_0(\beta,\beta)^{n+1} \leq M_0(\beta,\beta)^{t+1} \leq M_0(\beta,\beta)\mathrm{e}^{\ln(M_0(\beta,\beta))t}, \quad \text{for all} \quad t > 0$$

Note that as $M_0(\beta,\beta) \geq 1$ then $\omega(\beta) := \ln(M_0(\beta,\beta)) \geq 0$.

Now if $\alpha, \beta \in I$ and $\alpha \geq \beta$ and for $t > 1$ we have

$$\|S(t)\|_{\beta,\alpha} \leq \|S(t-1)\|_{\alpha,\alpha}\|S(1)\|_{\beta,\alpha} \leq M_0(\alpha,\alpha)\mathrm{e}^{\omega(\alpha)(t-1)}M_0(\beta,\alpha),$$

while for $0 < t < 1$ we have estimate (3.3). Then for any $\omega > \omega(\alpha)$ we get the result.
iii) First notice that from (3.3), for any $\alpha \geq \beta_0$, we have $\|S(1)\|_{\beta_0,\alpha} \leq M_0(\beta_0,\alpha)$. Now, if $t > 1$, then

$$\begin{aligned} \|S(t)u_0\|_\alpha &\leq \|S(1)\|_{\beta_0,\alpha}\|S(t-1)u_0\|_{\beta_0} \\ &\leq M_0(\beta_0,\alpha)M\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t}\|u_0\|_{\beta_0} \\ &\leq M_0(\beta_0,\alpha)\|i\|_{\alpha,\beta_0}M\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t}\|u_0\|_\alpha, \end{aligned}$$

where $\|i\|_{\alpha,\beta_0}$ denotes the norm of the inclusion $X^\alpha \hookrightarrow X^{\beta_0}$. Thus,

$$\|S(t)\|_{\alpha,\alpha} \leq K\mathrm{e}^{\omega_0 t}, \qquad \text{for all} \quad t > 1$$

with $K = M_0(\beta_0,\alpha)\|i\|_{\alpha,\beta_0}M\mathrm{e}^{-\omega_0}$.

On the other hand, if $\beta_0 \geq \alpha$, we also have, from (3.3), $\|S(1)\|_{\alpha,\beta_0} \leq M_0(\alpha,\beta_0)$ and for $t > 1$,

$$\begin{aligned} \|S(t)u_0\|_\alpha &\leq \|i\|_{\beta_0,\alpha}\|S(t)u_0\|_{\beta_0} \\ &\leq \|i\|_{\beta_0,\alpha}\|S(t-1)\|_{\beta_0,\beta_0}\|S(1)u_0\|_{\beta_0} \\ &\leq \|i\|_{\beta_0,\alpha}M\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t}\|S(1)\|_{\alpha,\beta_0}\|u_0\|_\alpha \\ &\leq \|i\|_{\beta_0,\alpha}M\mathrm{e}^{-\omega_0}M_0(\alpha,\beta_0)\mathrm{e}^{\omega_0 t}\|u_0\|_\alpha. \end{aligned}$$

Thus,

$$\|S(t)\|_{\alpha,\alpha} \leq K\mathrm{e}^{\omega_0 t}, \qquad \text{for all} \quad t > 1$$

with $K = M_0(\alpha,\beta_0)\|i\|_{\beta_0,\alpha}M\mathrm{e}^{-\omega_0}$.

Therefore, for any $\alpha \in I$, we have the estimate

$$\|S(t)\|_{\alpha,\alpha} \leq K(\alpha)\mathrm{e}^{\omega_0 t}, \qquad \text{for all} \quad t > 1.$$

Hence, again from (3.3) with $\beta = \alpha$, we get (3.6) with

$$M(\alpha) = \begin{cases} \max\{K(\alpha), M_0(\alpha,\alpha)\} & \text{if } \omega_0 \geq 0 \\ \max\{K(\alpha), M_0(\alpha,\alpha)\mathrm{e}^{-\omega_0}\} & \text{if } \omega_0 \leq 0. \end{cases}$$

If moreover for given $t_0 > 0$ we define $\delta = \|S(t_0)\|_{\beta_0,\beta_0}$ then for $t > 0$ we write $t = nt_0 + s$, with $n \in I\!N$ and $0 \leq s < t_0$. Then

$$\|S(t)\|_{\beta_0,\beta_0} \leq \delta^n \|S(s)\|_{\beta_0,\beta_0} \leq \mathrm{e}^{\ln(\delta)(\frac{t-s}{t_0})} M_0(\beta_0,\beta_0,t_0)$$

with $M_0(\beta_0,\beta_0,t_0)$ as in (3.4) and the result follows. In particular if $\delta < 1$ then $\omega_0 < 0$.
iv) Now note that if $0 < t \leq 1$, the estimate reduces to (3.3). On the other hand, if $t > 1$, then, using (3.3) and part iii), we get

$$\begin{aligned} \|S(t)\|_{\beta,\alpha} &\leq \|S(t-1)\|_{\alpha,\alpha}\|S(1)\|_{\beta,\alpha} \\ &\leq M_0(\beta,\alpha)M(\alpha)\mathrm{e}^{-\omega_0}\mathrm{e}^{\omega_0 t} = M_1(\beta,\alpha)\mathrm{e}^{\omega_0 t}. \end{aligned}$$

and the rest follows easily. ■

Remark 3.3 *Observe that if the original constants $M_0(\beta,\alpha)$ in (3.3), do not depend (or can be taken independent of $\alpha, \beta \in I$), then the same is true for $M_0(\beta,\alpha,T)$ and $M(\alpha)$ in (3.6) depends on the scale only through the norm of the inclusions $\|i\|_{\beta_0,\alpha}$ or $\|i\|_{\alpha,\beta_0}$.*

Hereafter we will make use extensively the following spaces.

Definition 3.4
For $T > 0$, $\gamma \in I$ and $\varepsilon \geq 0$ we define for functions in $L^\infty_{loc}((0,T], X^\gamma)$, the quantity

$$|||u|||_{\gamma,\varepsilon} = \sup_{t\in(0,T]} t^\varepsilon \|u(t)\|_\gamma$$

which becomes a norm on the set of functions where it is finite, that we denote $\mathcal{L}^\infty_\varepsilon((0,T], X^\gamma)$.

Note that this set always contains $L^\infty([0,T], X^\gamma)$ and coincides with it when $\varepsilon = 0$. Also, the spaces are increasing with ε. Then we have

Lemma 3.5 *For $T > 0$, $\gamma \in I$ and $\varepsilon \geq 0$, $\mathcal{L}^\infty_\varepsilon((0,T], X^\gamma)$ with norm $|||u|||_{\gamma,\varepsilon}$ is a Banach space.*

Proof. Note that $\{u^k\}_k$ is a Cauchy sequence in $\mathcal{L}^\infty_\varepsilon((0,T], X^\gamma)$ iff $v^k(t) = t^\varepsilon u^k(t)$ is a Cauchy sequence in $L^\infty([0,T], X^\gamma)$ and also $u^k(t)$ converges in X^γ to some $u(t)$ for each $t > 0$ and uniformly for $\delta \leq t \leq T$. The rest is easy. ■

Also, part i) in Lemma 3.2 can be stated as

Lemma 3.6 *Assume the semigroup $S(t)$ and the scale of spaces satisfy (3.3).*
Then, for any $\alpha, \beta \in I$ with $\alpha \geq \beta$ and $T > 0$,

$$S(\cdot) : X^{\beta} \longrightarrow \mathcal{L}^{\infty}_{\alpha-\beta}((0,T], X^{\alpha}), \qquad u_0 \mapsto S(\cdot)u_0$$

is linear and continuous.

Now we turn into the linear nonhomogeneous equation associated with a semigroup and a scale satisfying (3.3). Hence, assume $f \in L^1_{\mathrm{loc}}((0,T), X^{\gamma})$. Then we consider function defined by the variation of constants formula,

$$u(t; u_0) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)\, d\tau. \tag{3.7}$$

Then we prove the following result.

Theorem 3.7 *Assume $1 \leq \sigma \leq \infty$, $f \in L^{\sigma}((0,T), X^{\gamma})$, with $T > 0$, $u_0 \in X^{\gamma}$, and u is given by (3.7). Assume also*

$$0 \leq \gamma' - \gamma < \frac{1}{\sigma'},$$

(where $1/\sigma + 1/\sigma' = 1$), or $\gamma' = \gamma$ if $\sigma = 1$.
i) Then, $u \in \mathcal{L}^{\infty}_{\gamma'-\gamma}((0,T), X^{\gamma'})$ and

$$X^{\gamma} \times L^{\sigma}((0,T), X^{\gamma}) \ni (u_0, f) \longmapsto u \in \mathcal{L}^{\infty}_{\gamma'-\gamma}((0,T), X^{\gamma'})$$

is (linear and) continuous.
ii) Moreover, $u \in C((0,T], X^{\gamma'})$ and if $u_0 \in X^{\gamma'}$ then $u \in C([0,T], X^{\gamma'})$ and the mapping

$$X^{\gamma'} \times L^{\sigma}((0,T), X^{\gamma}) \ni (u_0, f) \longmapsto u \in C([0,T], X^{\gamma'})$$

is (linear and) continuous.
iii) Finally, if $f \in L^1((0,T), X^{\gamma})$ and is locally Lipschitz continuous, then $u(t)$ is a strong solution of

$$u_t + Au = f(t), \quad 0 < t \leq T, \qquad in \quad X^{\gamma}, \quad u(0) = u_0.$$

Proof.
i) Setting $u(t) = u(t; u_0)$, from (3.7) and (3.3), then for $\gamma' = \gamma$ if $\sigma = 1$ or for $0 \leq \gamma' - \gamma < \frac{1}{\sigma'}$ if $1 < \sigma \leq \infty$, we have

$$\|u(t)\|_{\gamma'} \leq M_0(T)\Big[t^{-(\gamma'-\gamma)}\|u_0\|_{\gamma} + \Big(\int_0^t (t-\tau)^{-\sigma'(\gamma'-\gamma)}\, d\tau\Big)^{1/\sigma'} \Big(\int_0^t \|f(\tau)\|^{\sigma}_{\gamma}\, d\tau\Big)^{1/\sigma}\Big]$$

which gives

$$\|u(t)\|_{\gamma'} \leq M_1(T)\Big[t^{-(\gamma'-\gamma)}\|u_0\|_{\gamma} + t^{\frac{1}{\sigma'}-(\gamma'-\gamma)} \Big(\int_0^t \|f(\tau)\|^{\sigma}_{\gamma}\, d\tau\Big)^{1/\sigma}\Big]$$

so it is bounded on finite intervals away from $t = 0$ and in particular $u(t) \in X^{\gamma'}$ for $t > 0$.

In particular

$$|||u|||_{\gamma',\gamma'-\gamma} \leq C(T)(\|u_0\|_\gamma + \|f\|_{L^\sigma((0,T),X^\gamma)})$$

which proves i).

ii) To prove continuity, fix $t > 0$ (or even $t = 0$ if $u_0 \in X^{\gamma'}$), $h > 0$, and then, from (3.7),

$$u(t+h) - u(t) = S(h)u(t) + \int_t^{t+h} S(t+h-\tau)f(\tau)\,d\tau$$

and (3.3) gives

$$\|u(t+h) - u(t)\|_{\gamma'} \leq \|S(h)u(t) - u(t)\|_{\gamma'} + M(T)\int_t^{t+h} (t+h-\tau)^{-(\gamma'-\gamma)}\|f(\tau)\|_\gamma\,d\tau.$$

Now the first term goes to zero as $h \to 0$ while the second term is bounded by

$$M\left(\int_t^{t+h} (t+h-\tau)^{-\sigma'(\gamma'-\gamma)}\,d\tau\right)^{1/\sigma'}\left(\int_t^{t+h} \|f(\tau)\|_\gamma^\sigma\,d\tau\right)^{1/\sigma} \leq M_0 h^{\frac{1}{\sigma'}-\gamma'-\gamma}$$

and we have continuity.

Moreover, if $u_0 \in X^{\gamma'}$, we have $\|u\|_{C([0,T],X^{\gamma'})} \leq C\left(\|u_0\|_{\gamma'} + \|f\|_{L^\sigma((0,T),X^\gamma)}\right)$ and this proves that the mapping $(u_0, f) \mapsto u$ is (linear and) continuous.

The proofs for $\sigma = \infty$ follows the same lines, with obvious modifications, and are therefore omitted.

Since the semigroup is analyitc in X^γ, part iii) follows from Lemma 3.2.1 in [11]. ■

Now, assume that for some fixed $\alpha \geq \beta$, with $0 \leq \alpha - \beta < 1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X^\alpha, X^\beta). \tag{3.8}$$

Consider the abstract linear integral problem, with u_0 to be chosen below

$$u(t; u_0) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau; u_0)\,d\tau, \qquad t > 0. \tag{3.9}$$

Definition 3.8 *For a given function u defined on $(0, T]$ and taking values in X^α, we define*

$$\mathcal{F}(u, u_0)(t) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau)\,d\tau, \qquad 0 < t \leq T \tag{3.10}$$

assumed it is well defined.

Then we have the following Lemma

Lemma 3.9 *Assume the semigroup $S(t)$ and the scale of spaces satisfy (3.3) and the perturbation P satisfies (3.8). Assume $\varepsilon \geq 0$, $\delta \geq 0$, $\gamma, \gamma' \in I$, with and $\gamma' \geq \gamma$, are such that*

$$\beta \leq \gamma' < \beta + 1 \quad \textit{and} \quad 0 \leq \varepsilon < 1 \tag{3.11}$$

Then for $u \in \mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ and $u_0 \in X^\gamma$, we have
i) For $0 < t \leq T$

$$t^\delta \| \int_0^t S(t-\tau) P u(\tau)\, d\tau \|_{\gamma'} \leq M_1(T) t^{\beta+\delta+1-\gamma'-\varepsilon} \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u|||_{\alpha,\varepsilon}$$

where $M_1(T) = c(\beta, \gamma', \varepsilon) M_0(\beta, \gamma', T)$.
ii) For $0 < t \leq T$

$$t^\delta \|\mathcal{F}(u, u_0)(t)\|_{\gamma'} \leq t^\delta \|S(t) u_0\|_{\gamma'} + M_1(T) t^{\beta+\delta+1-\gamma'-\varepsilon} \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u|||_{\alpha,\varepsilon}$$

with $M_1(T)$ as above.
iii) In particular, if

$$\delta = \gamma' - \gamma \geq 0 \quad \text{and} \quad \gamma < \beta + 1 - \varepsilon, \tag{3.12}$$

then

$$|||\mathcal{F}(u, u_0)|||_{\gamma',\delta} \leq |||S(\cdot) u_0|||_{\gamma',\delta} + C(T) \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T) T^{\beta+1-\gamma-\varepsilon}$ and all terms above are finite. In particular,

$$(u, u_0) \ni \mathcal{L}^\infty_\varepsilon((0,T], X^\alpha) \times X^\gamma \longmapsto \mathcal{F}(u, u_0) \in \mathcal{L}^\infty_{\gamma'-\gamma}((0,T], X^{\gamma'})$$

is linear and continuous.

Proof. We first prove part i), and then part ii) and iii) are immediate. Using (3.4) we have for $\gamma' \geq \beta$

$$t^\delta \| \int_0^t S(t-\tau) P u(\tau)\, d\tau \|_{\gamma'} \leq M(T) t^\delta \int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta}} \|P\|_{\alpha,\beta} \|u(\tau)\|_\alpha \, d\tau \leq$$

$$\leq M(T) |||u|||_{\alpha,\varepsilon} \|P\|_{\alpha,\beta} t^\delta \int_0^t \frac{1}{(t-\tau)^{\gamma'-\beta} \tau^\varepsilon}\, d\tau,$$

where we have set $M(T) = M_0(\beta, \gamma', T)$ as in (3.4). Now the change of variables $\tau = rt$ gives the result with

$$M_1(T) = M(T) (\int_0^1 \frac{1}{(1-r)^{\gamma'-\beta} r^\varepsilon}\, dr)$$

provided $\gamma' - \beta < 1$ and $\varepsilon < 1$ as in the statement. ■

Note that when we take $\gamma' > \gamma$ in Lemma 3.9 above, this result can be interpreted as a smoothing effect of the variation of constants formula (3.10). The same applies to the next result in which we analyze continuity in time.

Lemma 3.10 *With the same notations and assumptions as in Lemma 3.9, for $u \in \mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ and $u_0 \in X^\gamma$, if (3.11) holds, that is*

$$\beta \leq \gamma' < \beta + 1, \quad 0 \leq \varepsilon < 1$$

we have

$$\mathcal{F}(u, u_0) \in C((0,T], X^{\gamma'}).$$

Further more $\mathcal{F}(u, u_0)$ is locally Hölder continuous with values in $X^{\gamma'}$.

Proof. Fix $0 < t < T$ and take $h > 0$ small, so that $t + h \leq T$. Also take $0 < t^* < t - h$ to be chosen below. Then, from (3.10) we have

$$\mathcal{F}(u, u_0)(t^*) = S(t^*)u_0 + \int_0^{t^*} S(t^* - \tau)Pu(\tau)\, d\tau.$$

Then we get,

$$\mathcal{F}(u, u_0)(t + h) = S(t + h - t^*)\mathcal{F}(u, u_0)(t^*) + \int_{t^*}^{t+h} S(t + h - \tau)Pu(\tau)\, \mathrm{d}\tau,$$

$$\mathcal{F}(u, u_0)(t) = S(t - t^*)\mathcal{F}(u, u_0)(t^*) + \int_{t^*}^{t} S(t - \tau)Pu(\tau)\, \mathrm{d}\tau$$

The, suppressing temporarily the dependence in u_0, we get

$$\mathcal{F}(u)(t + h) - \mathcal{F}(u)(t) = \Big(S(t + h - t^*) - S(t - t^*)\Big)\mathcal{F}(u)(t^*)+$$

$$+ \int_t^{t+h} S(t + h - \tau)Pu(\tau)\, d\tau + \int_{t^*}^{t} \Big(S(h) - I\Big)S(t - \tau)Pu(\tau)\, d\tau. \tag{3.13}$$

Now we estimate in norm in (3.13) to get

$$\|\mathcal{F}(u)(t + h) - \mathcal{F}(u)(t)\|_{\gamma'} \leq \|\Big(S(t + h - t^*) - S(t - t^*)\Big)\mathcal{F}(u)(t^*)\|_{\gamma'}+$$

$$+M(T)\int_t^{t+h} (t+h-\tau)^{-(\gamma'-\beta)}\|P\|_{\alpha,\beta}\|u(\tau)\|_\alpha\, d\tau + M(T)\int_{t^*}^{t}(t-\tau)^{-(\gamma'-\beta)}\|P\|_{\alpha,\beta}\|u(\tau)\|_\alpha\, d\tau$$

where, in the third term, we have used that $\|S(h) - I\|_{\gamma',\gamma'}$ is bounded.

Now, since $S(t)$ is an analytic semigroup in $X^{\gamma'}$, the first term is bounded by a constant times h, while using that u is bounded in X^α on $[t^*, T]$, the second and third ones are bounded, respectively, by

$$K(T, u)\left(\int_t^{t+h} (t + h - \tau)^{-(\gamma'-\beta)}\, d\tau\right)\|P\|_{\mathcal{L}(X^\alpha,X^\beta)} = K_1(T, u)\|P\|_{\mathcal{L}(X^\alpha,X^\beta)}h^{1-(\gamma'-\beta)}$$

$$K(T, u)\left(\int_{t^*}^{t} (t - \tau)^{-(\gamma'-\beta)}\, d\tau\right)\|P\|_{\mathcal{L}(X^\alpha,X^\beta)} = K_1(T, u)\|P\|_{\mathcal{L}(X^\alpha,X^\beta)}(t - t^*)^{1-(\gamma'-\beta)}$$

Now taking $t^* = t - 2h$, we get the result. ■

Now we finally analyze continuity at $t = 0$.

Lemma 3.11 *With the notations of Lemma 3.9, if*

$$\beta \leq \gamma' < \beta + 1 - \varepsilon, \quad 0 \leq \varepsilon < 1$$

then for $u \in \mathcal{L}^\infty_\varepsilon((0, T], X^\alpha)$ *and* $u_0 \in X^{\gamma'}$

$$\mathcal{F}(u, u_0)(t) \to u_0, \quad in \quad X^{\gamma'}, \quad as \quad t \to 0.$$

Moreover, if the scale is nested and $u_0 \in X^\gamma$, *for some* $\gamma \leq \gamma'$,

$$\mathcal{F}(u, u_0)(t) \to u_0, \quad in \quad X^{\gamma}, \quad as \quad t \to 0.$$

Proof. By part i) in Lemma 3.9, with $\delta = 0$, we have

$$\| \int_0^t S(t-\tau)Pu(\tau)\, d\tau\|_{\gamma'} \leq M_1(T) t^{\beta+1-\gamma'-\varepsilon} \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u|||_{\alpha,\varepsilon}$$

where $M_1(T) = c(\beta, \gamma', \varepsilon) M_0(\beta, \gamma', T)$. Clearly the right hand side above goes to zero, as $t \to 0$.

On the other hand note that, by the choice of u_0 we have $S(t)u_0 \to u_0$ in $X^{\gamma'}$, or in X^γ when the spaces satisfy (3.2). ■

To find solutions of the linear problem (3.9), we start by the following "base" case.

Proposition 3.12 Solutions in X^α.

Assume the semigroup $S(t)$ and the scale of spaces satisfy (3.3) and assume also the perturbation satisfies (3.8). If

$$0 \leq \alpha - \beta < 1, \tag{3.14}$$

then for each $u_0 \in X^\alpha$ there exists a unique solution of (3.9), $u(\cdot; u_0) \in L^\infty_{loc}((0,\infty), X^\alpha)$, which is moreover in $C([0,\infty), X^\alpha)$.

Furthermore, for each $\alpha \leq \gamma' < \beta + 1$, we have that the solution satisfies

$$u(\cdot; u_0) \in C((0,\infty), X^{\gamma'}).$$

Even more, the unique solutions of (3.9) define a linear semigroup in X^α as

$$S_P(t)u_0 := u(t; u_0), \quad \textit{for all} \quad t > 0 \tag{3.15}$$

Proof. We show that there exists $T > 0$ such that $\mathcal{F}(\cdot, u_0)$ is a contraction in $L^\infty([0,T], X^\alpha)$. For this take $u_0 \in X^\alpha$ and u_1, u_2 in $L^\infty([0,T], X^\alpha)$ and note that, the right hand side of (3.10) is affine in u. Also from (3.14) we can use part iii) of Lemma 3.9 with $\gamma' = \gamma = \alpha$, $\delta = \varepsilon = 0$, to get $\mathcal{F}(u_i, u_0) \in L^\infty([0,T], X^\alpha)$ and also

$$|||\mathcal{F}(u_1, u_0) - \mathcal{F}(u_2, u_0)|||_{\alpha,0} \leq C(T) \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u_1 - u_2|||_{\alpha,0}$$

with $C(T) = M_1(T) T^{\beta+1-\alpha}$ and is a contraction for small enough T.

Since T can be taken independent of $u_0 \in X^\alpha$, it is easy to obtain that the solutions are defined for all $t \geq 0$. The continuity in time comes from Lemma 3.10 while the continuity at $t = 0$ in X^α comes from Lemma 3.11, with $\gamma' = \alpha$ and $\varepsilon = 0$.

Also, from (3.10) it follows that the operators defined in (3.15) are linear. Finally, the continuity of $S_P(t)$ in X^α will be proved in Proposition 3.15 below. ■

For weaker initial data we have the following result.

Theorem 3.13 Solutions in X^γ.

Assume the scale of spaces satisfy (3.3) and assume also the perturbation satisfies (3.8). If (3.14) is satisfied, that is

$$0 \leq \alpha - \beta < 1,$$

then for each

$$\alpha - 1 < \gamma \leq \alpha, \tag{3.16}$$

there exists T such that for each $u_0 \in X^\gamma$ there exists a unique solution of (3.9) $u \in \mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$, with $0 \leq \varepsilon = \alpha - \gamma < 1$.

Moreover the solution above is defined for all $t > 0$ and for each

$$\beta \leq \gamma' < \beta + 1, \quad \gamma' \geq \gamma, \tag{3.17}$$

we have that the solution satisfies

$$u(\cdot; u_0) \in C((0,\infty), X^{\gamma'}).$$

If, additionally $u_0 \in X^{\gamma'}$ then

$$u(\cdot; u_0) \in C([0,\infty), X^{\gamma'}).$$

Even more, for any $\gamma \in [\beta, \alpha]$, the unique solutions of (3.9) define a linear semigroup in X^γ as

$$S_P(t)u_0 := u(t; u_0), \quad \textit{for all} \quad t > s. \tag{3.18}$$

If the scale is nested, the same is true for any $\gamma \in (\alpha - 1, \alpha]$.

Proof. Now we show that $\mathcal{F}(\cdot, u_0)$ is a contraction in $\mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ with $0 \leq \varepsilon = \alpha - \gamma < 1$. For this take $u_0 \in X^\gamma$ and u_1, u_2 in $\mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ and note that the right hand side of (3.10) is affine in u. Also, from (3.14) and (3.16) we can use part iii) of Lemma 3.9 with $\gamma' = \alpha$ and $0 \leq \varepsilon = \delta = \alpha - \gamma < 1$, to get $\mathcal{F}(u_i, u_0) \in \mathcal{L}^\infty_\varepsilon((0,T], X^\alpha)$ and also

$$|||\mathcal{F}(u_1, u_0) - \mathcal{F}(u_2, u_0)|||_{\alpha,\varepsilon} \leq C(T) \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u_1 - u_2|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T) T^{\beta+1-\alpha}$ and is a contraction for small enough T.

Since T can be taken independent of $u_0 \in X^\gamma$, then it is easy to obtain that the solutions are defined for all $t \geq 0$.

The continuity in time comes from Lemma 3.10 while the continuity at $t = 0$ in $X^{\gamma'}$ comes from Lemma 3.11, with $\varepsilon = \alpha - \gamma$.

Now observe that in particular we have that for $t_0 > 0$ the solution satisfies $u(t_0) \in X^\alpha$ and $u \in L^\infty_{loc}([t_0, \infty), X^\alpha)$. Hence after time t_0, the solution coincides with the unique solution of Proposition 3.12.

If $\gamma \in [\beta, \alpha]$ then we can take $\gamma' = \gamma$. In particular, from this, it is easily seen that the linear operators $S_P(t)$ define a linear semigroup.

As before, the continuity of $S_P(t)$ in X^γ will be proved in Proposition 3.15 below. ■

Remark 3.14 *Note that the time T for which $\mathcal{F}$ is a contraction in Proposition 3.12 and in Theorem 3.13 can be taken the same for all perturbations such that*

$$\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0$$

for some $R_0 > 0$.

Now we prove the following estimates on the solutions of (3.9). In particular this proves that the semigroup $S_P(t)$ defined in (3.15) and (3.18) is continuous.

Proposition 3.15 *Assume (3.3), (3.8), and (3.14). Then for every $R_0 > 0$ and every*

$$P \in \mathcal{L}(X^\alpha, X^\beta) \quad \textit{with} \quad \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0$$

and for every $\gamma, \gamma' \in I$ such that

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma, \tag{3.19}$$

there exist constants $\omega = \omega(\gamma, \gamma', R_0) \geq 0$ and $M_0 = M_0(\gamma, \gamma', R_0)$ such that, for $t > 0$, the unique solution of (3.9) in Theorem 3.13 defines a mapping from X^γ into $X^{\gamma'}$

$$S_P(t)u_0 := u(t; u_0), \quad \textit{for all} \quad t > s$$

such that

$$\|S_P(t)u_0\|_{\gamma'} \leq M_0 \mathrm{e}^{\omega t} t^{-(\gamma' - \gamma)} \|u_0\|_\gamma, \quad \gamma' \geq \gamma. \tag{3.20}$$

In particular for any $\gamma \in [\beta, \alpha]$, $S_P(t) \in \mathcal{L}(X^\gamma)$ and it is a semigroup of linear continuous operators in X^γ.

The same is true for any $\gamma \in E(\alpha)$, if the scale is nested.

Proof. First, by (3.14) and (3.19), see (3.16), we can use part iii) in Lemma 3.9 for the fixed point of $\mathcal{F}$, with $\gamma' = \alpha$, $0 \leq \varepsilon = \delta = \alpha - \gamma < 1$, to get

$$|||u(\cdot; u_0)|||_{\alpha,\varepsilon} \leq |||S(\cdot)u_0|||_{\alpha,\varepsilon} + C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u(\cdot; u_0)|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T)T^{\beta+1-\alpha}$.

Then, note that, by (3.4) and the choice of ε, $|||S(\cdot)u_0|||_{\alpha,\varepsilon} \leq M_0(\gamma, \alpha, T)\|u_0\|_\gamma$ and by (3.8), take T such that $C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq \frac{1}{2}$ for all perturbations P as in the statement. Thus,

$$|||u(\cdot; u_0)|||_{\alpha,\varepsilon} \leq 2M_0(\gamma, \alpha, T)\|u_0\|_\gamma. \tag{3.21}$$

Now by (3.19), we can use part iii) in Lemma 3.9 for the fixed point of $\mathcal{F}$, with $\gamma' \geq \gamma$, $\delta = \gamma' - \gamma$, $0 \leq \varepsilon = \alpha - \gamma < 1$, to get

$$|||u(\cdot; u_0)|||_{\gamma',\delta} \leq |||S(\cdot)u_0|||_{\gamma',\delta} + C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} |||u(\cdot; u_0)|||_{\alpha,\varepsilon}.$$

again with $C(T) = M_1(T)T^{\beta+1-\alpha}$.

Then, note that, by (3.4) and the choice of δ, $|||S(\cdot)u_0|||_{\gamma',\delta} \leq M_0(\gamma, \gamma', T)\|u_0\|_\gamma$ and using (3.21), we have

$$|||u(\cdot; u_0)|||_{\gamma',\delta} \leq \Big(M_0(\gamma, \gamma', T) + C(T)\|P\|_{\mathcal{L}(X^\alpha, X^\beta)} 2M_0(\gamma, \alpha, T)\Big)\|u_0\|_\gamma.$$

Hence, by the choice of T above,

$$|||u(\cdot; u_0)|||_{\gamma',\delta} \leq \tilde{M}_0(\gamma, \gamma', T)\|u_0\|_\gamma$$

with $\tilde{M}_0(\gamma, \gamma', T) = M_0(\gamma, \gamma', T) + M_0(\gamma, \alpha, T)$.

Note that this gives,

$$\|S_P(t)\|_{\gamma,\gamma'} \leq \frac{\tilde{M}_0(\gamma, \gamma', T)}{t^{\gamma'-\gamma}}, \qquad \text{for all} \quad 0 < t \leq T. \tag{3.22}$$

Arguing as in part ii) in Lemma 3.2 we conclude (3.20).

In particular, for any $\gamma \in [\beta, \alpha]$, we can take $\gamma' = \gamma$ and from (3.20) we get that $S_P(t) \in \mathcal{L}(X^\gamma)$ and is a semigroup of linear continuous operators in X^γ. The same happens, from (3.20), for any $\gamma \in (\alpha - 1, \alpha]$ when the scale is nested. ■

Remark 3.16
i) Observe that if the original constants $M_0(\beta, \alpha)$ in (3.3), do not depend, or can be taken independent of $\alpha, \beta \in I$, then the same is true for $M_0(\gamma, \gamma', R_0)$ and $\omega(\gamma', R_0)$ in Proposition 3.15, which become independent of the spaces of the scale.
ii) If the scale is nested, once the perturbation P is fixed, the estimate (3.22) for $0 < t \leq 1$ allows to apply part iii) in Lemma 3.2 to obtain that there exists $\omega_0 = \omega_0(P)$ such that

$$\|S_P(t)u_0\|_\gamma \leq M_0(\gamma) e^{\omega_0 t} \|u_0\|_\gamma$$

for all $\gamma \in E(\alpha) = (\alpha - 1, \alpha]$. In turn, part iv) in Lemma 3.2 implies that (3.20) holds for some exponent independent of γ, γ'.

Remark 3.17 *Note that once the semigroup $S_P(t)$ is constructed as above we can use the results in parts i) and ii) in Theorem 3.7 for $f \in L^\sigma((0, T), X^\gamma)$, with $1 \leq \sigma \leq \infty$ and $\gamma \in E(\alpha) = (\alpha - 1, \alpha]$.*

Part iii) in Theorem 3.7 requires that the semigroup $S_P(t)$ is analytic. This can be achieved by means of Theorem 3.20 below.

Remark 3.18 Strong solutions
i) Note that for $u_0 \in X^\alpha$ the solution of (3.9) obtained in Proposition 3.12 satisfies

$$u(t; u_0) = S_P(t)u_0 = \mathcal{F}(u, u_0)(t) = S(t)u_0 + \int_0^t S(t - \tau) P u(\tau; u_0)\, d\tau.$$

Then, by Lemma 3.10, u is locally Hölder with values in X^α and then $h(\tau) = Pu(\tau)$ is locally Hölder with values in X^β (or in X^γ for any $\gamma \leq \beta$, if the scale is nested). Since $S(t)$ is analytic in X^β (or in X^γ for any $\gamma \leq \beta$, if the scale is nested), then Lemma 3.2.1 in [11] implies that, for $t > 0$, $u(t; u_0)$ is a C^1 strong solution of

$$u_t + Au = Pu, \qquad in \quad X^\beta,$$

(or in X^γ for any $\gamma \leq \beta$, if the scale is nested) where $-A$ is the infinitesimal generator of the semigroup $S(t)$ in X^β. In particular $-A + P$ is the infinitesimal generator of the semigroup $S_P(t)$.

For $u_0 \in X^\gamma$, for $\gamma \in E(\alpha)$, the solution of (3.9) obtained in Theorem 3.13 satisfies $u(t_0) \in X^\alpha$, for any $t_0 > 0$ and we can use the argument above for $t > t_0$ as well.

ii) Assume we can prove that the semigroup $S_P(t)$ is analytic in some X^γ for $\gamma \in E(\alpha)$. Then, thanks to Proposition 3.15, we can use the Transfer of Analyticity lemma, proved in [5]

Lemma 3.19 **Transfer of Analyticity**

Assume $\{S(t)\}_{t\geq 0}$ is an analytic semigroup in a Banach space X. Assume that for some Banach space Y and for $t > 0$,

$$S(t) \in \mathcal{L}(X, Y).$$

Then for each $u_0 \in X$, the curve of the semigroup $(0, \infty) \ni t \mapsto S(t)u_0$ is analytic in Y. Moreover for each t_0, the Taylor series in Y has a radius of convergence not smaller than the one in X.

In particular if $Y \subset X$, with continuous injection, then $\{S(t)\}_{t\geq 0}$ defines an analytic semigroup in Y.■

to conclude that the curves of the semigroup are analytic in $X^{\gamma'}$ for $\gamma' \in R(\beta)$, $\gamma' \geq \gamma$. In particular, if the scale is nested, we conclude that $S_P(t)$ defines an analytic semigroup in $X^{\gamma'}$ for $\gamma' \geq \gamma$.

Note that part ii) of Remark 3.18 rises the question of proving that the semigroup $S_P(t)$ is analytic in some X^γ for $\gamma \in E(\alpha)$. This can be achieved by some "elliptic" argument as below. Note that for this the scale and the semigroup $S(t)$ must satisfy a closer relationship than just (3.3); see part i) of Remark 3.1. Also the spaces in the scale must have some interpolation properties. This conditions are satisfied in many particular examples; see Sections 5 and 6.

Theorem 3.20 *Assume the scale is nested, that is, (3.2), and that for any $\gamma \in I$, if $-A$ denotes the infinitesimal generator of $S(t)$ in X^γ, then its domain is given by $D(A) = X^{\gamma+1}$.*

Also assume the scale satisfies the following interpolation property: if Y is a Banach space and $T \in \mathcal{L}(X^\gamma, Y)$ and $T \in \mathcal{L}(X^{\gamma'}, Y)$ then $T \in \mathcal{L}(X^{\theta\gamma+(1-\theta)\gamma'}, Y)$ for $\theta \in [0, 1]$ and

$$\|T\|_{\mathcal{L}(X^{\theta\gamma+(1-\theta)\gamma'}, Y)} \leq \|T\|^{\theta}_{\mathcal{L}(X^\gamma, Y)} \|T\|^{1-\theta}_{\mathcal{L}(X^{\gamma'}, Y)}.$$

As in Theorem 3.13 and Proposition 3.15, assume that for some fixed $\alpha \geq \beta$, with $0 \leq \alpha - \beta < 1$ we have a family of linear perturbations satisfying

$$P \in \mathcal{L}(X^\alpha, X^\beta) \quad \textit{with} \quad \|P\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0.$$

Then, there exists some $0 < \omega_0 = \omega_0(R_0)$ such that for any $Re(\lambda) \geq \omega_0$ and any $\gamma \in (\alpha - 1, \beta)$ the operator $A + \lambda I - P$, between $X^{\gamma+1}$ and X^γ, is invertible and

$$\|(A + \lambda I - P)^{-1}\|_{\mathcal{L}(X^\gamma, X^\gamma)} \leq \frac{C}{|\lambda|}, \qquad Re(\lambda) \geq \omega_0 \tag{3.23}$$

and

$$\|(A+\lambda I-P)^{-1}\|_{\mathcal{L}(X^{\gamma},X^{\gamma+1})}\leq C, \qquad Re(\lambda)\geq \omega_0 \tag{3.24}$$

where C is independent of P and λ.

In particular, for every $\gamma\in(\alpha-1,\beta)$, the semigroup $S_P(t)$ in X^{γ} in Theorem 3.13 is analytic.

The same holds for any $\gamma\in E(\alpha)=(\alpha-1,\alpha]$, by Lemma 3.19.

Proof. Note that for any $\gamma\in(\alpha-1,\beta)$ there exists $\beta\geq\tilde{\gamma}>\gamma$ such that

$$P: X^{\gamma+1}\subset X^{\alpha}\to X^{\beta}\subset X^{\tilde{\gamma}}\subset X^{\gamma}, \tag{3.25}$$

is linear and continuous and $\|P\|_{\mathcal{L}(X^{\gamma+1},X^{\tilde{\gamma}})}\leq \tilde{R}_0$.

Now for given $g\in X^{\gamma}$ the equation $Au+\lambda u-Pu=g$ can be written as

$$u=T_\lambda(u):=(A+\lambda I)^{-1}g+(A+\lambda I)^{-1}Pu.$$

Observe now that from the resolvent estimates in [3], Chapter I, Section 1.2, we have that for each $0\leq\alpha\leq 1$, for some $\omega>0$ and $C\geq 1$,

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X^{\gamma},X^{\gamma})}\leq \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda)\geq\omega$$

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X^{\gamma+1},X^{\gamma+1})}\leq \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda)\geq\omega$$

and

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X^{\gamma},X^{\gamma+1})}\leq C, \qquad \mathrm{Re}(\lambda)\geq\omega.$$

Interpolating these last two inequalities we get, for any $\gamma<\tilde{\gamma}<\gamma+1$

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X^{\tilde{\gamma}},X^{\gamma+1})}\leq \frac{C}{|\lambda|^{\tilde{\gamma}-\gamma}}.$$

Therefore, from this and (3.25) we get that the Lipschitz constant of $T_\lambda: X^{\gamma+1}\to X^{\gamma+1}$ is bounded by $\frac{C}{|\lambda|^{\gamma-\tilde{\gamma}}}$.

Therefore there exists $\omega_0\geq\omega$ such that T_λ is a contraction, with Lipschitz constant $\theta<1$ uniform for all $\mathrm{Re}(\lambda)\geq\omega_0$ and P. This implies that the unique fixed point of T_λ satisfies

$$\|u\|_{X^{\gamma+1}}\leq\frac{1}{1-\theta}\|(A+\lambda)^{-1}g\|_{X^{\gamma+1}}\leq\frac{C}{1-\theta}\|g\|_{X^{\gamma}}, \tag{3.26}$$

which proves (3.24). This, in turn, implies

$$\|u\|_{X^{\gamma}}\leq\|(A+\lambda)^{-1}g\|_{X^{\gamma}}+\|(A+\lambda)^{-1}Pu\|_{X^{\gamma}}\leq\frac{C}{|\lambda|}(\|g\|_{X^{\gamma}}+\|Pu\|_{X^{\gamma}})$$

and, using again (3.25) and (3.26), we get

$$\|u\|_{X^{\gamma}}\leq\frac{C}{|\lambda|}\|g\|_{X^{\gamma}}$$

which proves (3.23).

Then from e.g. Section 1.2 in Chapter I, in [3] we get that $-A+P$ generates an analytic semigroup $\tilde{S}(t)$ in X^γ and for every $u_0 \in X^\gamma$, the function $u(t;u_0) := \tilde{S}(t)u_0$ satisfies, for $t>0$, $u_t + Au = Pu$ in X^γ. Hence, $u(t;u_0)$ satisfies (3.9) and then by uniqueness we have $\tilde{S}(t)u_0 = S_P(t)u_0$.

The rest of the range for γ follows from Lemma 3.19. ■

Remark 3.21 *Note that the "elliptic" approach in the theorem above for the generation of an analytic semigroup gives worse results than the "parabolic" one used in Theorem 5.6. In fact, besides the extra assumptions on the scale, the semigroup in Theorem 3.20 is defined only in the spaces X^γ for $\gamma \in (\alpha-1,\beta)$, while in Theorem 3.13 the semigroup is defined for $\gamma \in (\alpha-1,\alpha]$.*

Now we consider the case in which several perturbations are considered sequentially. Assume

$$P^1, P^2 \in \mathcal{L}(X^\alpha, X^\beta), \quad \text{with} \quad 0 \le \alpha-\beta < 1$$

and consider the mapping $S_{P^1}(t)$ for $u_0 \in X^\gamma$ with $\gamma \in E(\alpha)$. Now we repeat the construction starting out of $S_{P^1}(t)$. Then we would have the new mapping that we denote $S_{[P^1,P^2]}(t)$ which is formally given by

$$S_{[P^1,P^2]}(t)u_0 = S_{P^1}(t)u_0 + \int_0^t S_{P^1}(t-\tau)P^2 S_{[P^1,P^2]}(\tau)u_0\, d\tau.$$

Now we state some properties of the resulting mappings.

Lemma 3.22 *i) If $P = aI$, with $a \in I\!R$, then*

$$S_{aI}(t) = \mathrm{e}^{at}S(t) \quad \text{in } X^\gamma \text{ for every } \gamma \in I.$$

ii) If $P \in \mathcal{L}(X^\alpha, X^\beta)$, $0 \le \alpha-\beta < 1$, and $a \in I\!R$ then

$$S_{[aI,P]}(t) = S_{[P,aI]}(t) = \mathrm{e}^{at}S_P(t) \quad \text{in } X^\gamma \text{ for every } \gamma \in E(\alpha).$$

If the scale is nested, or at least $X^\alpha \subset X^\beta$, then the operator above coincide with $S_{P+aI}(t)$.
iii) If $P^1, P^2 \in \mathcal{L}(X^\alpha, X^\beta)$, $0 \le \alpha-\beta < 1$, then

$$S_{[P^1,P^2]}(t) = S_{[P^2,P^1]}(t) = S_{P^1+P^2}(t) \quad \text{in } X^\gamma \text{ for every } \gamma \in [\beta,\alpha].$$

Proof.
i) Note that for $P = aI$ we can take $\alpha=\beta=\gamma$ for any $\gamma \in I$. Now for $u_0 \in X^\gamma$ we have that $u(t;u_0) = S_{aI}(t)u_0$ is the unique fixed point of (3.9), that is

$$u(t;u_0) = S(t)u_0 + a\int_0^t S(t-\tau)u(\tau;u_0)\, d\tau.$$

On the other hand, setting $v(t) = \mathrm{e}^{at} S(t) u_0$ we have

$$S(t)u_0 + a\int_0^t S(t-\tau)v(\tau)\,d\tau = \Big(1 + \int_0^t a\mathrm{e}^{a\tau}\,d\tau\Big)S(t)u_0 = \mathrm{e}^{at}S(t)u_0 = v(t).$$

Hence, $v(t) = u(t; u_0)$.
ii) From i), applied to $S_P(t)$, we have, for every $u_0 \in X^\gamma$ with $\gamma \in E(\alpha)$, $S_{[P,aI]}(t)u_0 = \mathrm{e}^{at} S_P(t) u_0$ which, by (3.9), can be written as

$$\mathrm{e}^{at}S_P(t)u_0 = \mathrm{e}^{at}S(t)u_0 + \int_0^t \mathrm{e}^{a(t-\tau)}S(t-\tau)P\Big(\mathrm{e}^{a\tau}S_P(\tau)u_0\Big)\,d\tau.$$

On the other hand, by the expression for $S_{aI}(t)$ from i), we have that for every $u_0 \in X^\gamma$ with $\gamma \in E(\alpha)$,

$$S_{[aI,P]}(t)u_0 = \mathrm{e}^{at}S(t)u_0 + \int_0^t \mathrm{e}^{a(t-\tau)}S(t-\tau)PS_{[aI,P]}(\tau)u_0\,d\tau.$$

The uniqueness of the fixed point problem gives

$$S_{[aI,P]}(t)u_0 = \mathrm{e}^{at}S_P(t)u_0 = S_{[P,aI]}(t)u_0.$$

The last part follows from iii) below.
iii) Note that from Remark 3.18, for every $\gamma \in E(\alpha)$ and $u_0 \in X^\gamma$, $u(t;u_0) := S_{P^1+P^2}(t)u_0$ satisfies, for $t > 0$,

$$u_t + Au = (P^1 + P^2)u, \qquad \text{in} \quad X^\beta,$$

(or in X^γ for any $\gamma \le \beta$, if the scale is nested) which can be written as

$$u_t + (A - P^1)u = P^2 u,$$

or as

$$u_t + (A - P^2)u = P^1 u.$$

In the first case we get,

$$u(t;u_0) = S_{P^1}(t-t_0)u(t_0;u_0) + \int_{t_0}^t S_{P^1}(t-\tau)P^2 u(\tau;u_0)\,d\tau$$

while in the second

$$u(t;u_0) = S_{P^2}(t-t_0)u(t_0;u_0) + \int_{t_0}^t S_{P^2}(t-\tau)P^1 u(\tau;u_0)\,d\tau$$

for $t > t_0$ and any $t_0 > 0$.

Now from Theorem 3.13, if $u_0 \in X^{\gamma'}$ with $\gamma' \in R(\beta)$ then $u(t_0;u_0) \to u_0$ in $X^{\gamma'}$ as $t_0 \to 0$. If $\gamma \in [\beta, \alpha]$ then we can take $\gamma' = \gamma$. From this and using again Theorem 3.13 we get that the first terms in the right hand side above converge to $S_{P^1}(t)u_0$ and $S_{P^2}(t)u_0$ in $X^{\gamma'}$ respectively. Hence in the first case we get

$$u(t;u_0) = S_{P^1}(t)u_0 + \int_0^t S_{P^1}(t-\tau)P^2 u(\tau;u_0)\,d\tau$$

while in the second

$$u(t;u_0) = S_{P^2}(t)u_0 + \int_0^t S_{P^2}(t-\tau)P^1 u(\tau;u_0)\, d\tau.$$

The uniqueness of solutions of (3.9) implies then that $u(t;u_0) = S_{[P^1,P^2]}(t)u_0$, in the first case and $u(t;u_0) = S_{[P^2,P^1]}(t)u_0$ in the second. ■

4 Convergence of linear semigroups

With the setting of Section 3, assume that we have two perturbations

$$P^i \in \mathcal{L}(X^\alpha, X^\beta), \quad i = 1,2, \quad 0 \le \alpha - \beta < 1.$$

Our goal is then to compare semigroups $S_{P^i}(t)$, $i = 1,2$. Hence assume

$$\|P^i\|_{\mathcal{L}(X^\alpha,X^\beta)} \le R_0 \quad i = 1,2$$

for some $R_0 > 0$. Also, consider the existence and regularity intervals as in (3.19)

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \ge \gamma.$$

Consider then two initial data $u_0^i \in X^\gamma$, $i = 1,2$ and the corresponding solution of (3.9)

$$u^i(t;u_0^i) = S_{P^i}(t)u_0^i = S(t)u_0^i + \int_0^t S(t-\tau)P^i u^i(\tau;u_0^i)\, d\tau, \qquad t > 0$$

and denote

$$z(t,u_0^1,u_0^2) = u^1(t;u_0^1) - u^2(t;u_0^2).$$

Theorem 4.1 *With the notations above, for any $R_0 > 0$,*
i) There exists a sufficiently small T_0 such that for all perturbations P^i such that $\|P^i\|_{\mathcal{L}(X^\alpha,X^\beta)} \le R_0$,

$$|||z(\cdot,u_0^1,u_0^2)|||_{\gamma',\delta} \le L(T_0,R_0)\Big(\|u_0^1 - u_0^2\|_\gamma + \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_\gamma\Big), \tag{4.1}$$

with $\delta = \gamma' - \gamma$. In particular,

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \le \frac{L(T_0,R_0)}{t^{\gamma'-\gamma}}\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \text{for all} \quad 0 < t \le T_0 \tag{4.2}$$

ii) For every $T > T_0$

$$\|z(t,u_0^1,u_0^2)\|_{\gamma'} \le M_2(T,T_0,R_0)\Big(\|u_0^1 - u_0^2\|_\gamma + \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_\gamma\Big), \quad T_0 \le t \le T. \tag{4.3}$$

In particular,

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \le L(T,T_0,R_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \text{for all} \quad T_0 < t \le T \tag{4.4}$$

Proof.
i) We first show the estimate for short times. Dropping momentarily the dependence in u_0^1, u_0^2, we get

$$z(t) = S(t)(u_0^1 - u_0^2) + \int_0^t S(t-\tau)\Big(P^1 - P^2\Big)u^2(\tau)\,d\tau + \int_0^t S(t-\tau)P^1 z(\tau)\,d\tau.$$

First note that by (3.20) in Proposition 3.15 we have, for $\varepsilon = \alpha - \gamma$ and for any $T > 0$,

$$|||u^i|||_{\alpha,\varepsilon} \leq M_0(T, R_0)\|u_0^i\|_\gamma. \tag{4.5}$$

Then, arguing as in Lemma 3.9, we get, with $\delta = \gamma' - \gamma$, $\varepsilon = \alpha - \gamma$

$$|||z|||_{\gamma',\delta} \leq |||S(\cdot)(u_0^1 - u_0^2)|||_{\gamma',\delta} + C(T)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}|||u^2|||_{\alpha,\varepsilon} + C(T)\|P^1\|_{\mathcal{L}(X^\alpha, X^\beta)}|||z|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T)T^{\beta+1-\alpha}$. Also note that the first term in the right hand side is bounded by $M(T)\|u_0^1 - u_0^2\|_\gamma$.

First, with $\gamma' = \alpha$, $\delta = \alpha - \gamma = \varepsilon$, we get

$$|||z|||_{\alpha,\varepsilon} \leq M(T)\|u_0^1 - u_0^2\|_\gamma + C(T)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}|||u^2|||_{\alpha,\varepsilon} + C(T)\|P^1\|_{\mathcal{L}(X^\alpha, X^\beta)}|||z|||_{\alpha,\varepsilon}$$

with $C(T) = M_1(T)T^{1+\beta-\alpha}$. Then for T_0 small such that $C(T_0)R_0 \leq 1/2$ we get

$$|||z|||_{\alpha,\varepsilon} \leq 2M(T_0)\|u_0^1 - u_0^2\|_\gamma + 2C(T_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}|||u^2|||_{\alpha,\varepsilon}. \tag{4.6}$$

Now with γ' and $\delta = \gamma' - \gamma$ and $\varepsilon = \alpha - \gamma$, we get

$$|||z|||_{\gamma',\delta} \leq M(T_0)\|u_0^1 - u_0^2\|_\gamma + C(T_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}|||u^2|||_{\alpha,\varepsilon} + C(T_0)\|P^1\|_{\mathcal{L}(X^\alpha, X^\beta)}|||z|||_{\alpha,\varepsilon}$$

again with $C(T_0) = M_1(T_0)T_0^{1+\beta-\alpha}$.

Hence, using (4.5) and (4.6), we get (4.1) which is valid for all P^i such that $\|P^i\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0$.

In particular, if $u_0^1 = u_0^2 = u_0$ then

$$|||z|||_{\gamma',\delta} \leq L(T_0, R_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}\|u_0\|_\gamma$$

which leads to (4.2).
ii) For $T_0 < t \leq T$ observe that

$$u^i(t; u_0^i) = S_{P^i}(t)u_0^i = S(t - T_0)u^i(T_0; u_0^i) + \int_{T_0}^t S(t-\tau)P^i u^i(\tau; u_0^i)\,\mathrm{d}\tau.$$

Dropping momentarily the dependence in u_0^1, u_0^2, we get

$$z(t) = S(t - T_0)z(T_0) + \int_{T_0}^t S(t-\tau)\Big(P^1 - P^2\Big)u^2(\tau)\,d\tau + \int_{T_0}^t S(t-\tau)P^1 z(\tau)\,d\tau.$$

and then

$$\|z(t)\|_{\gamma'} \le M(T)\|z(T_0)\|_{\gamma'} + K(T)\|P^1 - P^2\|_{\alpha,\beta}\int_{T_0}^{t}(t-\tau)^{-(\gamma'-\beta)}\|u^2(\tau)\|_{\alpha}\,d\tau +$$

$$+K(T)\|P^1\|_{\alpha,\beta}\int_{T_0}^{t}(t-\tau)^{-(\gamma'-\beta)}\|z(\tau)\|_{\alpha}\,d\tau.$$

Now, by (4.5), u^2 is bounded in X^α on $[T_0, T]$ and then the second term above is bounded by

$$K_1(T)\left(\int_{T_0}^{t}(t-\tau)^{-(\gamma'-\beta)}\,d\tau\right)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\sup_{[T_0,T]}\|u^2(t)\|_{\alpha}$$

which, using (4.5), is bounded by

$$K_2(T,T_0)\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_{\gamma}.$$

So we end up with

$$\|z(t)\|_{\gamma'} \le M(T)\|z(T_0)\|_{\gamma'} + K_2\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_{\gamma} + K_2\|P^1\|_{\alpha,\beta}\int_{T_0}^{t}(t-\tau)^{-(\gamma'-\beta)}\|z(\tau)\|_{\alpha}\,d\tau$$

for all $T_0 \le t \le T$.

Then using the singular Gronwall lemma, see Lemma 7.1.1, page 188, [11], we conclude

$$\|z(t)\|_{\gamma'} \le M_2(T)\Big(\|z(T_0)\|_{\gamma'} + \|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}\|u_0^2\|_{\gamma}\Big), \quad T_0 \le t \le T.$$

Using now the estimate for short times, (4.1), we get (4.3). In particular, if $u_0^1 = u_0^2 = u_0$ then we get (4.4). ■

Remark 4.2 *Observe that if both semigroups $S_{P^1}(t)$ and $S_{P^2}(t)$ decay exponentially, we actually get*

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \le \frac{L(R_0)e^{-\omega t}}{t^{\gamma'-\gamma}}\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \text{for all} \quad 0 < t < \infty$$

for some $\omega > 0$.

In the general case, if we replace $S(t)$ by $S_{-\lambda I}(t)$ with λ such that both $S_{[P^1,-\lambda I]}(t)$ and $S_{[P^2,-\lambda I]}(t)$ decay exponentially we get

$$\|S_{P^1}(t) - S_{P^2}(t)\|_{\gamma,\gamma'} \le \frac{L(R_0)e^{\omega t}}{t^{\gamma'-\gamma}}\|P^1 - P^2\|_{\mathcal{L}(X^\alpha,X^\beta)}, \qquad \text{for all} \quad 0 < t < \infty$$

for some $\omega \in \mathbb{R}$.

From the Theorem we get the following

Corollary 4.3
i) Given $P^1 \in \mathcal{L}(X^\alpha, X^\beta)$, assume for some $\gamma \in [\beta, \alpha]$, we have

$$\|S_{P^1}(t)\|_{\gamma,\gamma} \leq M e^{\omega_0 t}, \quad \text{for all} \quad t > 0 \tag{4.7}$$

for some $M = M(\gamma)$ and $\omega_0 \in I\!R$.

Then for any $\varepsilon > 0$, if $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ is sufficiently small we have

$$\|S_{P^2}(t)\|_{\gamma,\gamma} \leq M' e^{(\omega_0+\varepsilon)t}, \quad \text{for all} \quad t > 0$$

for some M' depending on M, ω_0, ε.

In particular, $S_{P^1}(t)$ decays exponentially, that is if $\omega_0 < 0$, then so does $S_{P^2}(t)$ if $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ is sufficiently small.

ii) If the scale is nested and (4.7) is satisfied for some $\gamma \in E(\alpha) = (\alpha-1, \alpha]$, then for any $\varepsilon > 0$, if $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ is sufficiently small we have for any $\gamma' \in E(\alpha) = (\alpha - 1, \alpha]$

$$\|S_{P^2}(t)\|_{\gamma',\gamma'} \leq M' e^{(\omega_0+\varepsilon)t}, \quad \text{for all} \quad t > 0$$

for some M' depending on $M, \omega_0, \gamma', \varepsilon$. Finally $S_{P^1}(t)$ and $S_{P^2}(t)$ satisfy the estimates (3.20) with $\omega = \omega_0 + \varepsilon$.

Proof.
i) Note that for $\varepsilon > 0$, $e^{-(\omega_0+\varepsilon)t} S_{P^1}(t) = S_{[P^1, -(\omega_0+\varepsilon)I]}(t)$ decays exponentially in X^γ. In particular there exists t_0 such that $\delta := \|S_{[P^1,-(\omega_0+\varepsilon)I]}(t_0)\|_{\gamma,\gamma} < 1$. Then, from Theorem 4.1, if $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ is sufficiently small we have $\delta' := \|S_{[P^2,-(\omega_0+\varepsilon)I]}(t_0)\|_{\gamma,\gamma} < 1$. Then the last part of part iii) in Lemma 3.2 implies that $e^{-(\omega_0+\varepsilon)t} S_{P^2}(t) = S_{[P^2,-(\omega_0+\varepsilon)I]}(t)$ decays exponentially in X^γ too and the result follows.
ii) When the scale is nested, observe that from part iii) in Lemma 3.2 we have that the exponential bounds for $S_{P^1}(t)$ and $S_{P^2}(t)$ are independent of γ; see (3.5) and (3.6). Therefore it is enough to have (4.7) for some $\gamma \in E(\alpha) = (\alpha - 1, \alpha]$.

The estimates (3.20) with $\omega = \omega_0 + \varepsilon$ follows from part iv) in Lemma 3.2. ■

5 Linear parabolic equations with nonsmooth coefficients

In this section we apply the abstract results in Sections 3 and 4 to the linear parabolic equations in Section 2 by considering perturbations with nonsmooth coefficients. For this recall that from (2.2) parabolic equations with smooth coefficients can be set in the Bessel potential scale

$$H^{2\gamma,q}_{\mathcal{B}}(\Omega) := X^\gamma, \quad \gamma \in I := [-1, 1]$$

with $1 < q < \infty$ fixed, which is a nested scale.

We now introduce some of the nonsmooth perturbations that we will consider for (2.1). Note that on the boundary we will perturb only the boundary condition on Γ.

Hence we define, for given functions m and m_0, the interior and boundary potential operators

$$< Q_0 u, \varphi >:= \int_\Omega m u \varphi, \qquad < R_0 u, \varphi >:= \int_\Gamma m_0 u \varphi \tag{5.1}$$

for suitable u and φ. Then we have

Lemma 5.1 *i) Assume that $m \in L^p(\Omega)$. Then for $s, \sigma \geq 0$ and*

$$s + \sigma \geq \frac{N}{p} \tag{5.2}$$

we have

$$Q_0 \in \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega)), \qquad \|Q_0\|_{\mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))} \leq C \|m\|_{L^p(\Omega)}.$$

ii) Assume $m_0 \in L^r(\Gamma)$. Then for $s > 1/q$, $\sigma > 1/q'$ and

$$s + \sigma \geq 1 + \frac{N-1}{r} \tag{5.3}$$

we have

$$R_0 \in \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega)), \qquad \|R_0\|_{\mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))} \leq C \|m_0\|_{L^r(\Gamma)}.$$

Proof. i) Note that for every $u \in H^{s,q}(\Omega)$ and $\varphi \in H^{\sigma,q'}(\Omega)$ we have

$$|\int_\Omega m u \varphi| \leq (\int_\Omega |m|^p)^{\frac{1}{p}} (\int_\Omega |u|^n)^{\frac{1}{n}} (\int_\Omega |\varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{p} + \frac{1}{n} + \frac{1}{\tau} = 1$. Using the sharp embedding of the Bessel spaces, we have

$$|\int_\Omega m u \varphi| \leq C \|m\|_{L^p(\Omega)} \|u\|_{H^{s,q}(\Omega)} \|\varphi\|_{H^{\sigma,q'}(\Omega)}$$

provided n, τ are such that $s - \frac{N}{q} \geq -\frac{N}{n}$, and $\sigma - \frac{N}{q'} \geq -\frac{N}{\tau}$. These conditions can be met because of (5.2).
ii) Now note that for every $u \in H^{s,q}(\Omega)$ and $\varphi \in H^{\sigma,q'}(\Omega)$ we have

$$|\int_\Gamma m_0 u \varphi| \leq (\int_\Gamma |m_0|^r)^{\frac{1}{r}} (\int_\Gamma |u|^n)^{\frac{1}{n}} (\int_\Gamma |\varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{r} + \frac{1}{n} + \frac{1}{\tau} = 1$. Using the trace properties of Bessel spaces we have

$$|\int_\Gamma m_0 u \varphi| \leq C \|m_0\|_{L^r(\Gamma)} \|u\|_{H^{s,q}(\Omega)} \|\varphi\|_{H^{\sigma,q'}(\Omega)}$$

provided n, τ are such that $s - \frac{N}{q} \geq -\frac{N-1}{n}$, with $s > \frac{1}{q}$, and $\sigma - \frac{N}{q'} \geq -\frac{N-1}{\tau}$, with $\sigma > \frac{1}{q'}$. These conditions can be met because of (5.3). $\blacksquare$

Now we define the first order perturbations. First the drift operator

$$< S_0 u, \varphi >:= \int_\Omega \vec{d} \nabla u \varphi, \tag{5.4}$$

and second, the divergence–0 operator

$$< T_0 u, \varphi > = < Div_0(\vec{d}u), \varphi >:= -\int_\Omega u \vec{d} \nabla \varphi, \tag{5.5}$$

for a given vector field $\vec{d}$.

Lemma 5.2 *Assume $\vec{d} \in L^\rho(\Omega)^N$.*
i) For $s \geq 1$, $\sigma \geq 0$ and

$$s + \sigma \geq 1 + \frac{N}{\rho} \tag{5.6}$$

the drift operator satisfies

$$S_0 \in \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega)), \qquad \|S_0\|_{\mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))} \leq C \|\vec{d}\|_{L^\rho(\Omega)^N}.$$

ii) For $s \geq 0$, $\sigma \geq 1$ and

$$s + \sigma \geq 1 + \frac{N}{\rho} \tag{5.7}$$

the divergence-0 operator satisfies

$$T_0 \in \mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega)), \qquad \|T_0\|_{\mathcal{L}(H^{s,q}(\Omega), H^{-\sigma,q}(\Omega))} \leq C \|\vec{d}\|_{L^\rho(\Omega)^N}.$$

Proof. i) Note that for every $u \in H^{s,q}(\Omega)$ and $\varphi \in H^{\sigma,q'}(\Omega)$ we have

$$|\int_\Omega \vec{d} \nabla u \varphi| \leq (\int_\Omega |\vec{d}|^\rho)^{\frac{1}{\rho}} (\int_\Omega |\nabla u|^n)^{\frac{1}{n}} (\int_\Omega |\varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{\rho} + \frac{1}{n} + \frac{1}{\tau} = 1$. Using the sharp embedding of the Bessel spaces, we have

$$|\int_\Omega \vec{d} \nabla u \varphi| \leq C \|\vec{d}\|_{L^\rho(\Omega)^N} \|u\|_{H^{s,q}(\Omega)} \|\varphi\|_{H^{\sigma,q'}(\Omega)}$$

provided n, τ are such that $s - \frac{N}{q} \geq 1 - \frac{N}{n}$, and $\sigma - \frac{N}{q'} \geq -\frac{N}{\tau}$. These conditions can be met because of (5.6).
ii) Now for every $u \in H^{s,q}(\Omega)$ and $\varphi \in H^{\sigma,q'}(\Omega)$ we have

$$|\int_\Omega u \vec{d} \nabla \varphi| \leq (\int_\Omega |\vec{d}|^\rho)^{\frac{1}{\rho}} (\int_\Omega |u|^n)^{\frac{1}{n}} (\int_\Omega |\nabla \varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{\rho} + \frac{1}{n} + \frac{1}{\tau} = 1$. Using the sharp embedding of the Bessel spaces, we have

$$|\int_\Omega \vec{d} \nabla u \varphi| \leq C \|\vec{d}\|_{L^\rho(\Omega)^N} \|u\|_{H^{s,q}(\Omega)} \|\varphi\|_{H^{\sigma,q'}(\Omega)}$$

provided n, τ are such that $s - \frac{N}{q} \geq 1 - \frac{N}{n}$, and $\sigma - \frac{N}{q'} \geq 1 - \frac{N}{\tau}$. These conditions can be met because of (5.7). ■

Using the embeddings of Bessel spaces in Section 2, we have

Corollary 5.3 *i) Assume that $m \in L^p(\Omega)$ for $p > N/2$. Then for $s, \sigma \geq 0$ and*

$$2 > s + \sigma \geq \frac{N}{p}$$

we have

$$Q_0 \in \mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega)), \qquad \|Q_0\|_{\mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega))} \leq C\|m\|_{L^p(\Omega)}.$$

ii) Assume $m_0 \in L^r(\Gamma)$ for $r > N-1$. Then for $s > 1/q$, $\sigma > 1/q'$ and

$$2 > s + \sigma \geq 1 + \frac{N-1}{r}$$

we have

$$R_0 \in \mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega)), \qquad \|R_0\|_{\mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega))} \leq C\|m_0\|_{L^r(\Gamma)}.$$

iii) Assume $\vec{d} \in L^\rho(\Omega)^N$, for $\rho > N$. Then for $s \geq 1$, $\sigma \geq 0$ and

$$2 > s + \sigma \geq 1 + \frac{N}{\rho}$$

the drift operator satisfies

$$S_0 \in \mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega)),, \qquad \|S_0\|_{\mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega))} \leq C\|\vec{d}\|_{L^\rho(\Omega)^N}.$$

iv) Assume $\vec{d} \in L^\rho(\Omega)^N$, for $\rho > N$. Then for $s \geq 0$, $\sigma \geq 1$ and

$$2 > s + \sigma \geq 1 + \frac{N}{\rho}$$

the divergence-0 operator satisfies

$$T_0 \in \mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega)), \qquad \|T_0\|_{\mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega))} \leq C\|\vec{d}\|_{L^\rho(\Omega)^N}.$$

Remark 5.4 *Observe that to define divergence operators, we have that, assuming regularity*

$$< Div(\vec{d}u), \varphi >:= \int_\Omega u\vec{d}\nabla\varphi + \int_{\partial\Omega} u\varphi\vec{d}\vec{n}.$$

If now u and φ are subjected to the boundary conditions of Section 2, we have that the boundary term above reduces to

$$\int_\Gamma u\varphi\vec{d}\vec{n}$$

for the case of Dirichlet boundary conditions. The case of Robin conditions corresponds to $\Gamma = \partial\Omega$.

In any case we have that, with the notations above

$$Div(\vec{d}u) = T_0(u) + R_0(u)$$

for some choice of boundary potential on $\partial\Omega$.

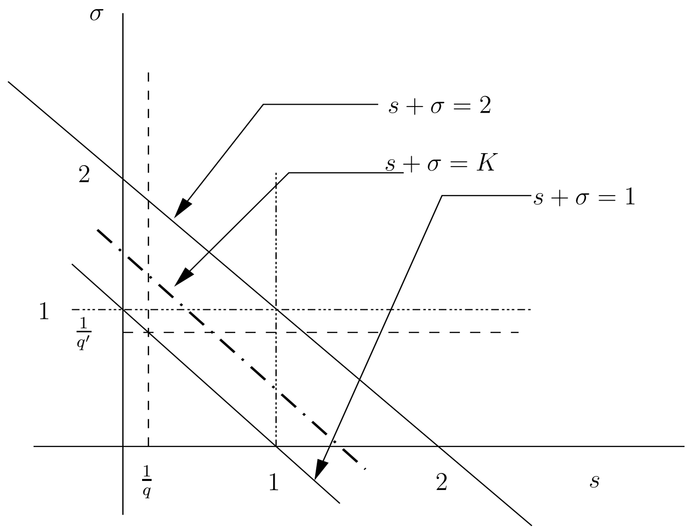

Figure 1: Regions for s and σ

Remark 5.5 *Observe that now we can consider perturbations P which are combinations of Q_0, R_0, S_0 and T_0, for which we have to restrict the ranges of s, σ in Corollary 5.3 depending on the combinations considered. Since the perturbation must satisfy (3.8) which translates here into $s + \sigma < 2$, S_0 and T_0 can not be combined together.*

Now we are ready to give the main results on the perturbed problems

$$\left\{ \begin{array}{rcll} u_t - \mathrm{div}(a(x)\nabla u) + c(x)u &=& m(x)u + \vec{d}(x)\nabla u & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& m_0(x)u & \text{on } \Gamma \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \setminus \Gamma \\ u(0) &=& u_0 & \text{in } \Omega \end{array} \right. \tag{5.8}$$

or

$$\left\{ \begin{array}{rcll} u_t - \mathrm{div}(a(x)\nabla u) + c(x)u &=& m(x)u + Div(\vec{d}(x)u) & \text{in } \Omega \\ a(x)\frac{\partial u}{\partial \vec{n}} + b(x)u &=& m_0(x)u & \text{on } \Gamma \\ \mathcal{B}u &=& 0 & \text{on } \partial\Omega \setminus \Gamma \\ u(0) &=& u_0 & \text{in } \Omega \end{array} \right. \tag{5.9}$$

with m in a bounded set in $L^p(\Omega)$, for $p > N/2$, m_0 in a bounded set in $L^r(\Gamma)$, for $r > N-1$ and $\vec{d}$ in a bounded set in $L^\rho(\Omega)^N$, for $\rho > N$.

Note that according to the remark above there are eleven kinds of perturbations that we will consider below, namely, four single perturbations:

$$P \text{ equals } Q_0,\ R_0,\ S_0 \text{ or } T_0; \tag{5.10}$$

five binary combinations:

$$P = Q_0 + R_0, \quad P = Q_0 + S_0, \quad P = Q_0 + T_0, \quad P = R_0 + S_0, \quad P = R_0 + T_0, \tag{5.11}$$

and two ternary combinations:

$$P = Q_0 + R_0 + T_0, \quad P = Q_0 + R_0 + S_0. \tag{5.12}$$

With this notations problems (5.8) or (5.9) can be all summarized in the weak formulation

$$\int_\Omega u_t\varphi + \int_\Omega a(x)\nabla u\nabla\varphi + \int_\Omega c(x)u\varphi + \int_{\partial\Omega} b(x)(x)u\varphi =< Pu, \varphi > \tag{5.13}$$

for all sufficiently smooth φ and where in the definition of P one must take into account (5.1), (5.4) and (5.5).

Therefore, applying the results in Section 3 we get

Theorem 5.6 *Assume that m is in a bounded set in $L^p(\Omega)$, with $p > N/2$, m_0 is in a bounded set in $L^r(\Gamma)$ and also $\vec{d}$ is in a bounded set in $L^\rho(\Omega)^N$, for $\rho > N$.*

Then, for any $1 < q < \infty$, and any P as in (5.10), (5.11) or (5.12) there exists and interval $I(q)$ (which depends on P too) containing $(-\frac{1}{2}, \frac{1}{2})$, such that we have a strongly continuous, order preserving, analytic semigroup, $S_P(t)$ in the space $H^{2\gamma,q}_{bc}(\Omega)$ for any $\gamma \in I(q)$.

Moreover the semigroup satisfies the smoothing estimates

$$\|S_P(t)u_0\|_{H^{2\gamma',q}_{bc}(\Omega)} \le \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega) \tag{5.14}$$

for every $\gamma, \gamma' \in I(q)$, with $\gamma' \ge \gamma$, for some $M_{\gamma',\gamma}$ and $\mu \in I\!R$ independent of P and $\gamma, \gamma' \in I(q)$. In particular, one has

$$\|S_P(t)u_0\|_{L^\tau(\Omega)} \le \frac{M_{\sigma,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\sigma}-\frac{1}{\tau})}}\|u_0\|_{L^\sigma(\Omega)}, \qquad t > 0, \quad u_0 \in L^\sigma(\Omega) \tag{5.15}$$

for $1 \le \sigma \le \tau \le \infty$ with $M_{\sigma,\tau}$ and μ independent of P.

Also, for every $u_0 \in H^{2\gamma,q}_{bc}(\Omega)$, with $\gamma \in I(q)$, the function $u(t; u_0) := S_P(t)u_0$ is $C^\nu(\overline{\Omega})$ for any $0 < \nu < 1$ and satisfies (5.13) for $t > 0$.

For each of the possible choices of P, the intervals $I(q)$ are given as follows

i) Single perturbations.

If P equals Q_0, then $I(q) = (-1, 1)$.
If P equals R_0, then $I(q) = (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$.
If P equals S_0, then $I(q) = (-\frac{1}{2}, 1)$.
If P equals T_0, then $I(q) = (-1, \frac{1}{2})$.

ii) Binary perturbations.

If $P = Q_0 + R_0$ then $I(q) = (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$.
If $P = Q_0 + S_0$ then $I(q) = (-\frac{1}{2}, 1)$.
If $P = Q_0 + T_0$ then $I(q) = (-1, \frac{1}{2})$.
If $P = R_0 + S_0$ then $I(q) = (-\frac{1}{2}, 1 - \frac{1}{2q'})$.
If $P = R_0 + T_0$ then $I(q) = (-1 + \frac{1}{2q}, \frac{1}{2})$.

iii) Ternary perturbations.

If $P = Q_0 + R_0 + T_0$ then $I(q) = (-1 + \frac{1}{2q}, \frac{1}{2})$.

If $P = Q_0 + R_0 + S_0$ then $I(q) = (-\frac{1}{2}, 1 - \frac{1}{2q'})$.

Proof. For any such perturbation P, by restricting s and σ according to Corollary 5.3 we get that

$$P \in \mathcal{L}(H^{s,q}_{bc}(\Omega), H^{-\sigma,q}_{bc}(\Omega)).$$

and is a bounded family in that space.

Then we can apply Theorem 3.13 and Proposition 3.15 with $\alpha = \frac{s}{2}$ and $\beta = \frac{-\sigma}{2}$ and we get the results in the statement for indexes

$$\gamma \in (\frac{s}{2} - 1, \frac{s}{2}], \qquad \gamma' \in [-\frac{\sigma}{2}, 1 - \frac{\sigma}{2}), \qquad \gamma' \geq \gamma.$$

Now note that as s, σ range over the set defined by the restrictions in Corollary 5.3, then the intervals for γ and γ' above fill some intervals which depended on the particular perturbation considered.

Note that all possible combinations, their restrictions, the ranges of s, σ and the resulting intervals for γ, γ' are as follows; see Figure 1. Note that all intervals above include $(-\frac{1}{2}, \frac{1}{2})$.

i) If P equals Q_0, the ranges for s, σ are $s \in [0, 2)$, $\sigma \in [0, 2)$ and then $\gamma, \gamma' \in (-1, 1)$.

If P equals R_0, the ranges for s, σ are $s \in (\frac{1}{q}, 2 - \frac{1}{q'})$, $\sigma \in (\frac{1}{q'}, 2 - \frac{1}{q})$ and then $\gamma, \gamma' \in (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$.

If P equals S_0, the ranges for s, σ are $s \in [1, 2)$, $\sigma \in (0, 1)$ and then $\gamma, \gamma' \in (-\frac{1}{2}, 1)$.

If P equals T_0, the ranges for s, σ are $s \in (0, 1)$, $\sigma \in [1, 2)$ and then $\gamma, \gamma' \in (-1, \frac{1}{2})$.

ii) Binary combinations. If $P = Q_0 + R_0$ the ranges for s, σ are $s \in (\frac{1}{q}, 2 - \frac{1}{q'})$, $\sigma \in (\frac{1}{q'}, 2 - \frac{1}{q})$ and then $\gamma, \gamma' \in (-1 + \frac{1}{2q}, 1 - \frac{1}{2q'})$.

If $P = Q_0 + S_0$ the ranges for s, σ are $s \in [1, 2)$, $\sigma \in (0, 1)$ and then $\gamma, \gamma' \in (-\frac{1}{2}, 1)$.

If $P = Q_0 + T_0$ the ranges for s, σ are $s \in (0, 1)$, $\sigma \in [1, 2)$ and then $\gamma, \gamma' \in (-1, \frac{1}{2})$.

If $P = R_0 + S_0$ the ranges for s, σ are $s \in [1, 2 - \frac{1}{q'})$, $\sigma \in (\frac{1}{q'}, 1)$ and then $\gamma, \gamma' \in (-\frac{1}{2}, 1 - \frac{1}{2q'})$.

If $P = R_0 + T_0$ the ranges for s, σ are $s \in (\frac{1}{q}, 1)$, $\sigma \in [1, 2 - \frac{1}{q})$ and then $\gamma, \gamma' \in (-1 + \frac{1}{2q}, \frac{1}{2})$.

iii) Ternary combinations. If $P = Q_0 + R_0 + T_0$ the ranges for s, σ are $s \in (\frac{1}{q}, 1)$, $\sigma \in [1, 2 - \frac{1}{q})$ and then $\gamma, \gamma' \in (-1 + \frac{1}{2q}, \frac{1}{2})$.

If $P = Q_0 + R_0 + S_0$ the ranges for s, σ are $s \in [1, 2 - \frac{1}{q'})$, $\sigma \in (\frac{1}{q'}, 1)$ and then $\gamma, \gamma' \in (-\frac{1}{2}, 1 - \frac{1}{2q'})$.

With all these, we get (5.14). That μ in the exponential bound of the semigroup is independent of γ, γ' follows from Lemma 3.2.

Now we prove (5.15). Note that starting with out of $1 < q < \infty$ and $\gamma = 0$, taking $\gamma' \in (0, \frac{1}{2})$ and using the sharp embeddings of Bessel spaces in Section 2 we get (5.15) with $\sigma = q$ and $q < \tau < \frac{Nq}{N-q}$. Iterating this argument, starting out of τ and $\gamma = 0$ we get (5.15).

That $u(t; u_0) := S_P(t)u_0$ satisfies (5.13) follows from part i) in Remark 3.18 while the analyticity of the semigroup is a consequence of Theorem 3.20.

The Hölder regularity of $u(t; u_0) := S_P(t)u_0$ follows by observing that by (5.14) and (5.15) the solution enters $H^{2\gamma',\sigma}_{bc}(\Omega)$ for all $1 < \sigma < \infty$ and any $\gamma' \in I(\sigma)$. Since this interval is at least $(-\frac{1}{2}, \frac{1}{2})$, again the embeddings of Bessel potential spaces gives that the solution is $C^\nu(\overline{\Omega})$ for any $0 < \nu < 1$.

Finally, that the semigroup $S_P(t)$ is order preserving follows from Theorem 5.9 below. In fact for C^1 smooth m, m_0 and $\vec{d}$ the results in [2] imply that the semigroups are order preserving. Then the convergence in Theorem 5.9 below shows the same property for the non smooth case. ■

Remark 5.7 *Note that in fact we can get that for any $\tilde{q} > q > 1$*

$$\|S_P(t)u_0\|_{H^{2\gamma',\tilde{q}}_{bc}(\Omega)} \leq \frac{Me^{\mu t}}{t^{\gamma'-\gamma+\frac{N}{2}(\frac{1}{q}-\frac{1}{\tilde{q}})}}\|u_0\|_{H^{2\gamma,q}_{bc}(\Omega)}, \qquad t > 0, \quad u_0 \in H^{2\gamma,q}_{bc}(\Omega)$$

and for any $\gamma, \gamma' \in (-\frac{1}{2}, \frac{1}{2})$, $\gamma' > \gamma$.

Remark 5.8 *After Theorem 5.6 we can apply the results in Theorem 3.7 for the linear nonhomogeneous problem with $f \in L^\sigma((0,T), H^{2\gamma,q}_{bc}(\Omega))$ with $1 \leq \sigma \leq \infty$ and $\gamma \in I(q)$.*

Now applying the results in Section 4 we have

Theorem 5.9 *With the notations in Theorem 5.6, assume*

$$m_\varepsilon \to m \quad \text{in } L^p(\Omega), \quad p > \frac{N}{2},$$
$$m_{0,\varepsilon} \to m_0 \quad \text{in } L^r(\Gamma), \quad r > N-1,$$
$$\vec{d}_\varepsilon \to \vec{d} \quad \text{in } L^\rho(\Omega)^N, \quad \rho > N.$$

and for any $1 < q < \infty$, consider the corresponding semigroups $S_{P_\varepsilon}(t)$ and $S_{P_0}(t)$.

Then for every

$$\gamma, \gamma' \in I(q), \quad \gamma' \geq \gamma,$$

and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_{P_0}(t)\|_{\mathcal{L}(H^{2\gamma,q}_{bc}(\Omega), H^{2\gamma',q}_{bc}(\Omega))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \text{for all} \quad 0 < t \leq T. \tag{5.16}$$

In particular, for any $0 < \nu < 1$ the solutions $u^\varepsilon(t; u_0) := S_{P_\varepsilon}(t)u_0$ converge to solutions $u(t; u_0) := S_P(t)u_0$ in $C^\nu(\overline{\Omega})$ uniformly in bounded time intervals away from $t = 0$.

Proof. Most of the statement is a direct application of Theorem 4.1, using that the perturbation $P_\varepsilon \to P_0$ in $\mathcal{L}(X^\alpha, X^\beta)$.

The Hölder convergence of solutions follows by a bootstrap argument, based on (5.14), (5.15) and (5.16) which proves convergence in $H^{2\gamma',\sigma}_{bc}(\Omega)$ for any $1 < \sigma < \infty$ and $\gamma' \in (0, \frac{1}{2})$ on bounded time intervals away from $t = 0$. ■

6 Parabolic equations in unbounded domains

We consider now the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } I\!\!R^N, \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{6.1}$$

whose solution is given by

$$u(t,x) = S_0(t)u_0 = (4\pi t)^{-N/2} \int_{I\!\!R^N} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy \tag{6.2}$$

in different settings as explained below. Observe that all the results below hold for more general operators in divergence form and with bounded coefficients that have Gaussian bounds on the fundamental kernel, see [9].

6.1 Lebesgue scale

By elementary properties of convolution it is known that the heat equation in $I\!\!R^N$ satisfies

$$\|S_0(t)u_0\|_{L^r(I\!\!R^N)} \leq \frac{M_{r,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(I\!\!R^N)}, \qquad t > 0, \quad u_0 \in L^q(I\!\!R^N)$$

for $1 \leq q \leq r \leq \infty$ and $\mu_0 > 0$ arbitrary. This holds for more general operators in divergence form and with bounded coefficients that have Gaussian bounds on the fundamental kernel, see [9].

Then for $1 \leq q \leq \infty$, we denote

$$L^q(I\!\!R^N) := X^{\gamma(q)}, \quad \gamma = \frac{-N}{2q} \in I := [-N/2, 0], \tag{6.3}$$

(which is not a nested scale) and we have again (3.3).

Then as a consequence of Hölder's inequality we have,

Lemma 6.1 *Assume that $m \in L^p(I\!\!R^N)$, with $p > N/2$ then the multiplication operator*

$$Q_0 u(x) = m(x)u(x),$$

satisfies for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$

$$Q_0 \in \mathcal{L}(L^r(I\!\!R^N), L^s(I\!\!R^N)), \qquad \|Q_0\|_{\mathcal{L}(L^r(I\!\!R^N), L^s(I\!\!R^N))} \leq C\|m\|_{L^p(I\!\!R^N)}.$$

Then, using Theorem 3.13, Proposition 3.15 and Theorem 4.1 we get

Theorem 6.2

i) Assume that m is in a bounded set in $L^p(\mathbb{R}^N)$, with $p > N/2$. Then for any $1 \leq q < \infty$ the Schrödinger equation

$$\left\{ \begin{array}{rcll} u_t - \Delta u & = & m(x)u & \text{in } \mathbb{R}^N, \quad t > 0 \\ u(0) & = & u_0 & \text{in } \mathbb{R}^N \end{array} \right.$$

defines an order preserving, analytic semigroup $S_m(t)$ in $L^q(\mathbb{R}^N)$ that satisfies

$$\|S_m(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{r,q}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in L^q(\mathbb{R}^N)$$

for $1 \leq q \leq r \leq \infty$, with $M_{q,r}$ and μ independent of m.
ii) If

$$m_\varepsilon \to m \quad \text{in } L^p(\mathbb{R}^N), \quad p > \frac{N}{2},$$

then for every $1 \leq q \leq r \leq \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{m_\varepsilon}(t) - S_m(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}, \qquad \text{for all} \quad 0 < t \leq T.$$

Proof. With the notations in Lemma 6.1 and according to (6.3) we have for each $\alpha_0 := \frac{-N}{2p'} \leq \alpha \leq 0$

$$Q_0 \in \mathcal{L}(X^\alpha, X^\beta), \qquad \|Q_0\|_{\mathcal{L}(X^\alpha,X^\beta)} \leq C\|m\|_{L^p(\mathbb{R}^N)}$$

with $\alpha = \frac{-N}{2r}$, $\beta = \frac{-N}{2s}$ and $0 \leq \alpha - \beta = \frac{-N}{2r} + \frac{N}{2s} = \frac{N}{2p} < 1$.

Hence, we can apply Theorem 3.13 and Proposition 3.15 and we get a semigroup in X^γ for $\gamma \in [\beta, \alpha]$ and the smoothing estimates for for indexes

$$\gamma \in (\alpha - 1, \alpha], \qquad \gamma' \in [\beta, \beta + 1), \qquad \gamma' \geq \gamma.$$

As α runs the interval $\alpha_0 := \frac{-N}{2p'} \leq \alpha \leq 0$, and noting that for $\alpha = \alpha_0$ we have $\beta = -N/2$ it is clear that γ and γ' fill the interval I.

The second part is a direct consequence of Theorem 4.1. Also this proves that the semigroup is order preserving.

Note that the analyticity of the semigroup above will result from the result in the next subsection. ■

Remark 6.3 *After the theorem above, we can apply the results in Theorem 3.7 for $f \in L^\sigma((0,T), L^q(\mathbb{R}^N))$ with $1 \leq \sigma \leq \infty$ and $1 \leq q < \infty$.*

6.2 Bessel scale

Sharper results on nonsmooth perturbations of the heat equation (6.1) can be obtained using a setting in the nested Bessel scale $H^{2\alpha,q}(I\!R^N)$ for fixed $1 < q < \infty$ and $\alpha \in [-1,1]$; see [2]. Note that now the embeddings in Section 2, $H^{2\alpha,q}(I\!R^N) \subset L^r(I\!R^N)$ hold only for $r \geq q$. Just as for the case of a bounded domain in Section 2, the results in [2] imply (6.1) defines an analytic semigroup in th nested scale $X^\alpha = H^{2\alpha,q}(I\!R^N)$, $\alpha \in I := [-1,1]$ and satisfies the smoothing estimates

$$\|S_0(t)u_0\|_{H^{2\alpha,q}(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in H^{2\beta,q}(I\!R^N)$$

with $\mu_0 > 0$ arbitrary.

In order to introduce the class of perturbations we will consider below, we define, for $1 \leq p < \infty$, the *uniform space* $L^p_U(I\!R^N)$ as the set of functions $\phi \in L^p_{loc}(I\!R^N)$ such that

$$\sup_{x\in\mathbb{R}^N} \int_{B(x,1)} |\phi(y)|^p \, dy < \infty \tag{6.4}$$

with norm

$$\|\phi\|_{L^p_U(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{L^p(B(x,1))}.$$

Observe that for $p = \infty$, using the analogous definition, we have $L^\infty_U(I\!R^N) = L^\infty(I\!R^N)$ with norm $\|\phi\|_{L^\infty_U(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N} \|\phi\|_{L^\infty(B(x,1))} = \|\phi\|_{L^\infty(\mathbb{R}^N)}$. Observe that $L^p_U(I\!R^N)$ contains $L^\infty(I\!R^N)$, $L^r(I\!R^N)$ and $L^r_U(I\!R^N)$ for any $r \geq p$.

Then we have

Lemma 6.4 *i) Assume that $m \in L^p_U(I\!R^N)$, with $p > N/2$ then for $s \geq 0$, $\sigma \geq 0$ and*

$$2 > s + \sigma \geq \frac{N}{p}$$

the multiplication operator

$$Q_0 u(x) = m(x)u(x),$$

satisfies

$$Q_0 \in \mathcal{L}(H^{s,q}(I\!R^N), H^{-\sigma,q}(I\!R^N)), \qquad \|Q_0\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N),H^{-\sigma,q}(\mathbb{R}^N))} \leq C\|m\|_{L^p(\mathbb{R}^N)}.$$

ii) Assume $\vec{d} \in L^\rho_U(I\!R^N)^N$ with $\rho > N$. Then for $s \geq 1$, $\sigma \geq 0$ and

$$2 > s + \sigma \geq 1 + \frac{N}{\rho}$$

the drift operator defined as

$$< S_0 u, \varphi >:= \int_{\mathbb{R}^N} \vec{d}\nabla u \varphi,$$

satisfies

$$S_0 \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{-\sigma,q}(\mathbb{R}^N)), \qquad \|S_0\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N),H^{-\sigma,q}(\mathbb{R}^N))} \leq C\|\vec{d}\|_{L^\rho(\mathbb{R}^N)^N}.$$

iii) Assume $\vec{d} \in L^\rho_U(\mathbb{R}^N)^N$ *with* $\rho > N$*. Then for* $s \geq 0$*,* $\sigma \geq 1$ *and*

$$2 > s + \sigma \geq 1 + \frac{N}{\rho}$$

the divergence-0 operator defined as

$$< T_0 u, \varphi > = < Div_0(\vec{d}u), \varphi > := -\int_{\mathbb{R}^N} u\vec{d}\nabla\varphi,$$

satisfies

$$T_0 \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{-\sigma,q}(\mathbb{R}^N)), \qquad \|T_0\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N),H^{-\sigma,q}(\mathbb{R}^N))} \leq C\|\vec{d}\|_{L^\rho(\mathbb{R}^N)^N}.$$

Proof.
i) Denote by $\{Q_i\}$ the family of cubes centered at points of integer coordinates in $\mathbb{R}^N$ and with edges of length 1 parallel to the axes. Thus since $m \in L^p_U(\mathbb{R}^N)$, for every $u \in H^{s,q}(\mathbb{R}^N)$ and $\varphi \in H^{\sigma,q'}(\mathbb{R}^N)$ we have

$$|\int_{\mathbb{R}^N} mu\varphi| \leq \sum_i |\int_{Q_i} mu\varphi| \leq \sum_i (\int_{Q_i} |m|^p)^{\frac{1}{p}} (\int_{Q_i} |u|^n)^{\frac{1}{n}} (\int_{Q_i} |\varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{p} + \frac{1}{n} + \frac{1}{\tau} = 1$. Using the sharp embedding of the Bessel spaces in Q_i, we have

$$|\int_{\mathbb{R}^N} mu\varphi| \leq C\|m\|_{L^p_U(\mathbb{R}^N)} \sum_i \|u\|_{H^{s,q}(Q_i)} \|\varphi\|_{H^{\sigma,q'}(Q_i)}$$

with constants independent of i, provided n, τ are such that $s - \frac{N}{q} \geq -\frac{N}{n}$, and $\sigma - \frac{N}{q'} \geq -\frac{N}{\tau}$ and $n \geq q$, $\tau \geq q'$.

These conditions can be met because of the restrictions in the statement.

To conclude the proof note that in Lemma 2.4 in [6] it was proved that for any $1 < q < \infty$ and $0 \leq \alpha \leq 1$

$$\sum_i \|\phi\|^q_{H^{2\alpha,q}(Q_i)} \leq C\|\phi\|^q_{H^{2\alpha,q}(\mathbb{R}^N)} \qquad \text{for all} \quad \phi \in H^{2\alpha,q}(\mathbb{R}^N).$$

Hence Hölder's inequality for sequences, gives

$$|\int_{\mathbb{R}^N} mu\varphi| \leq C\|m\|_{L^p_U(\mathbb{R}^N)} \|u\|_{H^{s,q}(\mathbb{R}^N)} \|\varphi\|_{H^{\sigma,q'}(\mathbb{R}^N)}.$$

ii) Note that for every $u \in H^{s,q}(\mathbb{R}^N)$ and $\varphi \in H^{\sigma,q'}(\mathbb{R}^N)$ we have

$$|\int_{\mathbb{R}^N} \vec{d}\nabla u\varphi| \leq \sum_i (\int_{Q_i} |\vec{d}|^\rho)^{\frac{1}{\rho}} (\int_{Q_i} |\nabla u|^n)^{\frac{1}{n}} (\int_{Q_i} |\varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{\rho}+\frac{1}{n}+\frac{1}{\tau}=1$. Using the sharp embedding of the Bessel spaces in Q_i, we have

$$|\int_{\mathbf{R}^N} \vec{d}\nabla u\varphi| \leq C\|\vec{d}\|_{L^\rho_U(\mathbf{R}^N)^N} \sum_i \|u\|_{H^{s,q}(Q_i)}\|\varphi\|_{H^{\sigma,q'}(Q_i)}$$

with constants independent of i, provided n,τ are such that $s-\frac{N}{q}\geq 1-\frac{N}{n}$, and $\sigma-\frac{N}{q'}\geq -\frac{N}{\tau}$ and $n\geq q$, $\tau\geq q'$.

These conditions can be met because of the restrictions in the statement. As before we get

$$|\int_{\mathbf{R}^N} \vec{d}\nabla u\varphi| \leq C\|\vec{d}\|_{L^\rho_U(\mathbf{R}^N)^N}\|u\|_{H^{s,q}(\mathbf{R}^N)}\|\varphi\|_{H^{\sigma,q'}(\mathbf{R}^N)}.$$

iii) Now for every $u\in H^{s,q}(\Omega)$ and $\varphi\in H^{\sigma,q'}(\Omega)$ we have

$$|\int_{\mathbf{R}^N} u\vec{d}\nabla\varphi| \leq \sum_i (\int_{Q_i}|\vec{d}|^\rho)^{\frac{1}{\rho}}(\int_{Q_i}|u|^n)^{\frac{1}{n}}(\int_{Q_i}|\nabla\varphi|^\tau)^{\frac{1}{\tau}}$$

where $\frac{1}{\rho}+\frac{1}{n}+\frac{1}{\tau}=1$. Using the sharp embedding of the Bessel spaces in Q_i, we have

$$|\int_{\mathbf{R}^N} u\vec{d}\nabla\varphi| \leq C\|\vec{d}\|_{L^\rho_U(\mathbf{R}^N)^N} \sum_i \|u\|_{H^{s,q}(Q_i)}\|\varphi\|_{H^{\sigma,q'}(Q_i)}$$

with constants independent of i, provided n,τ are such that $s-\frac{N}{q}\geq 1-\frac{N}{n}$, and $\sigma-\frac{N}{q'}\geq 1-\frac{N}{\tau}$ and $n\geq q$, $\tau\geq q'$.

These conditions can be met because of the restrictions in the statement. As before we get

$$|\int_{\mathbf{R}^N} u\vec{d}\nabla\varphi| \leq C\|\vec{d}\|_{L^\rho_U(\mathbf{R}^N)^N}\|u\|_{H^{s,q}(\mathbf{R}^N)}\|\varphi\|_{H^{\sigma,q'}(\mathbf{R}^N)}.$$

■

Observe that as in Section 5, S_0 and T_0 can not be combined together. Hence, with the notations of previous subsections, we now can consider the perturbations:

$$P \quad \text{equals} \quad Q_0,\ S_0 \text{ or } T_0 \tag{6.5}$$

or the binary perturbations

$$P=Q_0+S_0, \qquad P=Q_0+T_0. \tag{6.6}$$

In a completely analogous way to Theorem 5.6 we get

Theorem 6.5 *Assume that m is in a bounded set in $L^p_U(\mathbb{R}^N)$, with $p>N/2$, and $\vec{d}$ is in a bounded set in $L^\rho_U(\mathbb{R}^N)^N$, for $\rho>N$.*

Then, for any $1<q<\infty$, and any P as in (6.5), (6.6) there exists and interval $I(q)$ (which depends on P too) that contains $(-\frac{1}{2},\frac{1}{2})$, such that we have a strongly continuous, order preserving, analytic semigroup, $S_P(t)$ in the space $H^{2\gamma,q}(\mathbb{R}^N)$ for any $\gamma\in I(q)$.

Moreover the semigroup satisfies the smoothing estimates

$$\|S_P(t)u_0\|_{H^{2\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{2\gamma,q}(\mathbb{R}^N)}, \qquad t>0, \quad u_0 \in H^{2\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q)$*, with* $\gamma' \geq \gamma$*, for some* $M_{\gamma',\gamma}$ *and* $\mu \in \mathbb{R}$ *independent of* P *and* $\gamma, \gamma' \in I(q)$*. In particular, one has*

$$\|S_P(t)u_0\|_{L^\tau(\mathbb{R}^N)} \leq \frac{M_{\sigma,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\sigma}-\frac{1}{\tau})}}\|u_0\|_{L^\sigma(\mathbb{R}^N)}, \qquad t>0, \quad u_0 \in L^\sigma(\mathbb{R}^N)$$

for $1 \leq \sigma \leq \tau \leq \infty$ *with* $M_{\sigma,\tau}$ *and* μ *independent of* P*.*

Also, for every $u_0 \in H^{2\gamma,q}(\mathbb{R}^N)$*, with* $\gamma \in I(q)$*, the function* $u(t;u_0) := S_P(t)u_0$ *is* $C^\nu_b(\mathbb{R}^N)$ *for any* $0<\nu<1$ *and satisfies*

$$\int_{\mathbb{R}^N} u_t\varphi + \int_{\mathbb{R}^N} \nabla u \nabla\varphi = < Pu, \varphi >$$

for sufficiently smooth φ *and* $t>0$*.*

For each of the possible choices of P*, the intervals* $I(q)$ *are given as follows.*

i) Single perturbations.

If P *equals* Q_0*, then* $I(q) = (-1,1)$*.*

If P *equals* S_0*, then* $I(q) = (-\frac{1}{2},1)$*.*

If P *equals* T_0*, then* $I(q) = (-1,\frac{1}{2})$*.*

ii) Binary perturbations.

If $P = Q_0 + S_0$ *then* $I(q) = (-\frac{1}{2},1)$*. If* $P = Q_0 + T_0$ *then* $I(q) = (-1,\frac{1}{2})$*.*

Note that the analyticity of the semigroup is obtained from Theorem 3.20 while the order preserving property follows from the order preserving from Proposition 5.3 in [5].

Remark 6.6 *After the theorem above we can apply the results in Theorem 3.7 for* $f \in L^\sigma((0,T), H^{2\gamma,q}(\mathbb{R}^N))$ *with* $1 \leq \sigma \leq \infty$ *and* $\gamma \in I(q)$*.*

Now applying the results in Section 4 and in an analogous way to Theorem 5.9 we have

Theorem 6.7 *With the notations above assume*

$$m_\varepsilon \to m \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{2},$$

$$\vec{d}_\varepsilon \to \vec{d} \quad \text{in } L^\rho_U(\mathbb{R}^N)^N, \quad \rho > N.$$

and for any $1<q<\infty$*, consider the corresponding semigroups* $S_{P_\varepsilon}(t)$ *and* $S_{P_0}(t)$*.*

Then for every

$$\gamma, \gamma' \in I(q), \quad \gamma' \geq \gamma,$$

and $T>0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$*, such that*

$$\|S_{P_\varepsilon}(t) - S_{P_0}(t)\|_{\mathcal{L}(H^{2\gamma,q}(\mathbb{R}^N), H^{2\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \text{for all} \quad 0<t\leq T.$$

In particular, for any $0<\nu<1$ *the solutions* $u^\varepsilon(t;u_0) := S_{P_\varepsilon}(t)u_0$ *converge to solutions* $u(t;u_0) := S_P(t)u_0$ *in* $C^\nu_b(\mathbb{R}^N)$ *uniformly in bounded time intervals away from* $t=0$*.*

6.3 Locally uniform spaces

The heat equation (6.1) and its perturbations can also be considered in much larger spaces, by taking initial data in locally uniform spaces.

For this consider the locally uniform space $L^q_U(I\!R^N)$ for $1 \leq q \leq \infty$ as in (6.4) and denote by $\dot{L}^q_U(I\!R^N)$ the closed subspace of $L^q_U(I\!R^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{L^q_U(\mathbb{R}^N)}$, that is

$$\|\tau_y\phi - \phi\|_{L^q_U(\mathbb{R}^N)} \to 0 \quad \text{as} \quad |y| \to 0$$

where $\{\tau_y, y \in I\!R^N\}$ denotes the group of translations. Note that $L^q(I\!R^N) \subset \dot{L}^q_U(I\!R^N)$ for $1 \leq q < \infty$ and for $q = \infty$ we get $\dot{L}^\infty_U(I\!R^N) = BUC(I\!R^N)$.

Using these spaces and (6.2) it was proved in Proposition 2.1 and Theorem 2.1 in [5] that the heat equation defines an order preserving analytic semigroup in $L^q_U(I\!R^N)$, for $1 \leq q < \infty$, which is strongly continuous in $\dot{L}^q_U(I\!R^N)$ and satisfies

$$\|S_0(t)u_0\|_{L^r_U(\mathbb{R}^N)} \leq \frac{M_{r,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in L^q_U(I\!R^N)$$

and

$$\|S_0(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{r,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{L}^q_U(I\!R^N)$$

for $1 \leq q \leq r \leq \infty$ for $\mu_0 > 0$ arbitrary. This holds for more general operators in divergence form and with bounded coefficients that have Gaussian bounds on the fundamental kernel, see [9] and Theorem 2.3 in [5].

Then for $1 \leq q \leq \infty$, we denote

$$\dot{L}^q_U(I\!R^N) := X^{\gamma(q)}, \quad \gamma = \frac{-N}{2q} \in I := [-N/2, 0], \tag{6.7}$$

which is a nested scale and we have again (3.3).

Then we have

Lemma 6.8 *i) Assume that $m \in L^p_U(I\!R^N)$, with $p > N/2$ then the multiplication operator*

$$Q_0u(x) = m(x)u(x),$$

satisfies for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$

$$Q_0 \in \mathcal{L}(L^r_U(I\!R^N), L^s_U(I\!R^N)), \qquad \|Q_0\|_{\mathcal{L}(L^r_U(\mathbb{R}^N), L^s_U(\mathbb{R}^N))} \leq \|m\|_{L^p_U(\mathbb{R}^N)}$$

ii) If moreover $m \in \dot{L}^p_U(I\!R^N)$, with $p > N/2$ we have for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$

$$Q_0 \in \mathcal{L}(\dot{L}^r_U(I\!R^N), \dot{L}^s_U(I\!R^N)), \qquad \|Q_0\|_{\mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N))} \leq \|m\|_{L^p_U(\mathbb{R}^N)}.$$

Proof. For any $x_0 \in I\!R^N$, from Hölder's inequality we have, for any $r \geq p'$,

$$\|Q_0 u\|_{L^s(B(x_0,1))} \leq \|m\|_{L^p(B(x_0,1))} \|u\|_{L^r(B(x_0,1))}, \quad \text{for} \quad \frac{1}{s} = \frac{1}{r} + \frac{1}{p}$$

which gives part i).

Now for every $y \in I\!R^N$ we have

$$\|\tau_y(Q_0 u) - Q_0 u\|_{L^s(B(x_0,1))} \leq \|\tau_y m\|_{L^p(B(x_0,1))} \|\tau_y u - u\|_{L^r(B(x_0,1))} +$$

$$+ \|\tau_y m - m\|_{L^p(B(x_0,1))} \|u\|_{L^r(B(x_0,1))}.$$

Hence, we get part ii). ■

Then, using Theorem 3.13, Proposition 3.15 and Theorem 4.1 we get

Theorem 6.9
i) Assume that m is in a bounded set in $\dot{L}^p_U(I\!R^N)$, with $p > N/2$. Then for any $1 \leq q < \infty$ the Schrödinger equation

$$\left\{ \begin{array}{rcll} u_t - \Delta u & = & m(x) u & \text{in } I\!R^N, \quad t > 0 \\ u(0) & = & u_0 & \text{in } I\!R^N \end{array} \right.$$

defines an order preserving analytic semigroup $S_m(t)$ in $\dot{L}^q_U(I\!R^N)$ that satisfies

$$\|S_m(t) u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{r,q} e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})}} \|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{L}^q_U(I\!R^N)$$

for $1 \leq q \leq r \leq \infty$, with $M_{q,r}$ and μ independent of m.
ii) If

$$m_\varepsilon \to m \quad \text{in } \dot{L}^p_U(I\!R^N), \quad p > \frac{N}{2},$$

then for every

$$1 \leq q \leq r \leq \infty$$

and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{m_\varepsilon}(t) - S_m(t)\|_{\mathcal{L}(\dot{L}^q_U(\mathbb{R}^N), \dot{L}^r_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{2}(\frac{1}{q} - \frac{1}{r})}}, \qquad \text{for all} \quad 0 < t \leq T.$$

Proof. Just note that according to (6.7) we read Lemma 6.8 as $Q_0 \in \mathcal{L}(X^\alpha, X^\beta)$, $\alpha = \frac{-N}{2r}$, $\beta = \frac{-N}{2s}$, for any $0 \geq \alpha \geq \alpha_0$, with $0 \leq \alpha - \beta = \frac{-N}{2r} + \frac{N}{2s} = \frac{N}{2p} < 1$ and $\|Q_0\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq C\|m\|_{\dot{L}^p_U(\mathbb{R}^N)}$. ■

Remark 6.10 *Observe that part i) of this result recovers the estimates in Proposition 3.2 in [5].*

Analyticity follows from Theorem 6.12 below, and the order preserving from Proposition 5.3 in [5].

Remark 6.11 *After the theorem above we can apply the results in Theorem 3.7 for* $f \in L^{\sigma}((0,T), \dot{L}^q_U(\mathbb{R}^N))$ *with* $1 \leq \sigma \leq \infty$ *and* $1 \leq q < \infty$.

In order to obtain sharper results and to consider drift perturbations to the heat equation we introduce the *uniform Bessel-Sobolev spaces* $H^{k,q}_U(\mathbb{R}^N)$, with $k \in \mathbb{N}$, as the set of functions $\phi \in H^{k,q}_{loc}(\mathbb{R}^N)$ such that

$$\|\phi\|_{H^{k,q}_U(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \|\phi\|_{H^{k,q}(B(x,1))} < \infty$$

for $k \in \mathbb{N}$. Then denote by $\dot{H}^{k,q}_U(\mathbb{R}^N)$ a subspace of $H^{k,q}_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{H^{k,q}_U(\mathbb{R}^N)}$, that is

$$\|\tau_y \phi - \phi\|_{H^{k,q}_U(\mathbb{R}^N)} \to 0 \ \text{ as } \ |y| \to 0$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations.

Consider the complex interpolation functor denoted by $[\ ,\ ]_\theta$, for $\theta \in (0,1)$, [7, 14]. Then for $1 \leq q < \infty$, $k \in \mathbb{N} \cup \{0\}$ and $s \in (k, k+1)$ we define $\theta \in (0,1)$ such that $s = \theta(1+k) + (1-\theta)k$, that is $\theta = s - k$. Then one can define the intermediate spaces as

$$H^{s,q}_U(\mathbb{R}^N) = [H^{k+1,q}_U(\mathbb{R}^N), H^{k,q}_U(\mathbb{R}^N)]_\theta,$$

and

$$\dot{H}^{s,q}_U(\mathbb{R}^N) = [\dot{H}^{k+1,q}_U(\mathbb{R}^N), \dot{H}^{k,q}_U(\mathbb{R}^N)]_\theta.$$

The following results is a simplified version of Theorem 5.3 in [5], which applies to more general operators. It was proved using purely elliptic arguments.

Theorem 6.12 *Assume* $1 < q < \infty$, $m \in \dot{L}^p_U(\mathbb{R}^N)$ *and* $\vec{d} \in \dot{L}^\rho_U(\mathbb{R}^N)^N$ *satisfy that* $\rho = q$ *if* $q > N$ *or* $\rho > N$ *otherwise, and* $p = q$ *if* $q > N/2$ *or* $p > N/2$ *otherwise.*

Define the elliptic operator $\dot{L}^q_U(\mathbb{R}^N)$,

$$Au = -\Delta u + \vec{d}(x)\nabla u + m(x)u$$

with domain $D(A) = \dot{H}^{2,q}_U(\mathbb{R}^N)$.

Then, $-A$*, generates a strongly continuous analytic semigroup on* $\dot{L}^q_U(\mathbb{R}^N)$ *and the associated fractional power spaces are given by*

$$\dot{H}^{2\alpha,q}_U(\mathbb{R}^N) = [\dot{H}^{2,q}_U(\mathbb{R}^N), \dot{L}^q_U(\mathbb{R}^N)]_\alpha, \quad \alpha \in [0,1]. \blacksquare$$

In particular, Theorem 6.12 implies that for every $1 < q < \infty$, the heat equation (6.1) defines and order preserving analytic semigroup in the nested scale $X^\alpha = \dot{H}^{2\alpha,q}_U(\mathbb{R}^N)$, $0 \leq \alpha < 1$ with

$$\|S_0(t)u_0\|_{\dot{H}^{2\alpha,q}_U(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{2\beta,q}_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{H}^{2\beta,q}(\mathbb{R}^N)$$

with $\mu_0 > 0$ arbitrary, for any $0 \leq \alpha \leq \beta < 1$.

Then we have

Lemma 6.13 *i) Assume that* $m \in \dot{L}^p_U(I\!R^N)$*, with* $p > N/2$ *and let* $1 < q \leq p$*. Then the multiplication operator*

$$Q_0 u(x) = m(x)u(x),$$

satisfies

$$Q_0 \in \mathcal{L}(\dot{H}^{2\alpha,q}_U(I\!R^N), \dot{L}^q_U(I\!R^N)), \qquad \|Q_0\|_{\mathcal{L}(\dot{H}^{\alpha,q}_U(\boldsymbol{R}^N), \dot{L}^q_U(\boldsymbol{R}^N))} \leq C\|m\|_{\dot{L}^p_U(\boldsymbol{R}^N)}$$

for $\alpha_0 := \frac{N}{2p} \leq \alpha < 1$.
ii) Assume $\vec{d} \in \dot{L}^\rho_U(I\!R^N)^N$*, for* $\rho > N$ *and let* $1 < q \leq \rho$*. Then the drift operator*

$$S_0 u(x) = \vec{d}(x)\nabla u(x)$$

satisfies

$$S_0 \in \mathcal{L}(\dot{H}^{2\gamma,q}_U(I\!R^N), \dot{L}^q_U(I\!R^N)), \qquad \|S_0\|_{\mathcal{L}(\dot{H}^{2\gamma,q}_U(\boldsymbol{R}^N), \dot{L}^q_U(\boldsymbol{R}^N))} \leq C\|\vec{d}\|_{\dot{L}^\rho_U(\boldsymbol{R}^N)^N}$$

for $\gamma_0 := \frac{1}{2} + \frac{N}{2\rho} \leq \gamma < 1$.

Proof. Note that, as in [5], but with a different notation, we can also define $W^{s,q}_U(I\!R^N)$ as the space of functions such that

$$\sup_{y \in \boldsymbol{R}^N} \|\phi\|_{H^{s,q}(B(y,1))} < \infty$$

with norm

$$\|\phi\|_{W^{s,q}_U(\boldsymbol{R}^N)} = \sup_{y \in \boldsymbol{R}^N} \|\phi\|_{H^{s,q}(B(y,1))}$$

and consider the subset of elements which are translational continuous, $\dot{W}^{s,q}_U(I\!R^N)$. With this definition, we have $H^{s,q}_U(I\!R^N) \subset W^{s,q}_U(I\!R^N)$ and $\dot{H}^{s,q}_U(I\!R^N) \subset \dot{W}^{s,q}_U(I\!R^N)$; see Proposition 4.2 in [5].

With this it is easy to see that the sharp embeddings of Bessel spaces in Section 2 translate into

$$\dot{H}^{s,q}_U(I\!R^N) \subset \begin{cases} \dot{L}^r_U(I\!R^N), \ s - \frac{N}{q} \geq -\frac{N}{r}, \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} < 0 \\ \dot{L}^r_U(I\!R^N), \ \ 1 \leq r < \infty, & \text{if } s - \frac{N}{q} = 0 \\ C^\eta_b(I\!R^N) & \text{if } s - \frac{N}{q} > \eta > 0. \end{cases}$$

Arguing as in Lemma 6.8 we get that

$$Q_0 \in \mathcal{L}(\dot{L}^r_U(I\!R^N), \dot{L}^q_U(I\!R^N)), \quad \text{for} \quad \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$$

and now chose α such that $\dot{H}^{2\alpha,q}_U(I\!R^N) \subset \dot{L}^r_U(I\!R^N)$ which leads to $2\alpha \geq \frac{N}{p}$.

On the other hand, from the argument above its clear that for $q \leq \rho$ and $2\alpha \geq \frac{N}{\rho}$ we have

$$S_0 \in \mathcal{L}(\dot{H}^{2\alpha+1,q}_U(I\!R^N), \dot{L}^q_U(I\!R^N)),$$

and setting $2\gamma = 2\alpha + 1$ we get the result. ■

Note that since we have no results on the negative part of the scale of uniform Bessel spaces, we can not handle the divergence operator. Hence, we now can consider the perturbations:

$$P \quad \text{equals} \quad Q_0 \text{ or } S_0 \tag{6.8}$$

or the binary perturbation

$$P = Q_0 + S_0. \tag{6.9}$$

Hence using the previous results we get

Theorem 6.14 *Assume that m is in a bounded set in $\dot{L}^p_U(\mathbb{R}^N)$, with $p > N/2$, and $\vec{d}$ is in a bounded set in $\dot{L}^\rho_U(\mathbb{R}^N)^N$, for $\rho > N$ and define $q_0 = \min\{p, \rho\} > N/2$.*
i) Then, for any $1 < q \leq q_0$, and any P as in (6.8), (6.9) we have a strongly continuous, order preserving, analytic semigroup, $S_P(t)$ in the space $\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$ for any $\gamma \in [0, 1)$.

Moreover the semigroup satisfies the smoothing estimates

$$\|S_P(t)u_0\|_{\dot{H}^{2\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{\dot{H}^{2\gamma,q}_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in [0, 1)$, with $\gamma' \geq \gamma$, for some $M_{\gamma',\gamma}$ and $\mu \in \mathbb{R}$ independent of P and $\gamma, \gamma' \in [0, 1)$. In particular, one has

$$\|S_P(t)u_0\|_{\dot{L}^\tau_U(\mathbb{R}^N)} \leq \frac{M_{\sigma,\tau}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{\sigma}-\frac{1}{\tau})}}\|u_0\|_{\dot{L}^\sigma_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{L}^\sigma_U(\mathbb{R}^N)$$

for $1 \leq \sigma \leq \tau \leq \infty$ with $M_{\sigma,\tau}$ and μ independent of P.

Also, for every $u_0 \in \dot{H}^{2\gamma,q}_U(\mathbb{R}^N)$, with $\gamma \in [0, 1)$, the function $u(t; u_0) := S_P(t)u_0$ is $C^\nu_b(\mathbb{R}^N)$ for some $0 < \nu < 1$ (or any $0 < \nu < 1$ if $q_0 > N$) and satisfies

$$u_t - \Delta u = Pu \qquad \text{in } \mathbb{R}^N \text{ for } t > 0.$$

ii) Assume

$$m_\varepsilon \to m \quad \text{in } \dot{L}^p_U(\mathbb{R}^N), \quad p > \frac{N}{2},$$
$$\vec{d_\varepsilon} \to \vec{d} \quad \text{in } \dot{L}^\rho_U(\mathbb{R}^N)^N, \quad \rho > N.$$

and for any $1 < q \leq q_0$, consider the corresponding semigroups $S_{P_\varepsilon}(t)$ and $S_{P_0}(t)$.

Then for every $\gamma, \gamma' \in [0, 1)$, $\gamma' \geq \gamma$, and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_{P_0}(t)\|_{\mathcal{L}(\dot{H}^{2\gamma,q}_U(\mathbb{R}^N),\dot{H}^{2\gamma',q}_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \text{for all} \quad 0 < t \leq T.$$

In particular, for some $0 < \nu < 1$ (or any $0 < \nu < 1$ if $q_0 > N$), the solutions $u^\varepsilon(t; u_0) := S_{P_\varepsilon}(t)u_0$ converge to solutions $u(t; u_0) := S_P(t)u_0$ in $C^\nu_b(\mathbb{R}^N)$ uniformly in bounded time intervals away from $t = 0$.

Proof. From Lemma 6.13, taking $\alpha = \frac{N}{2p} < 1$ for $P = Q_0$, $\alpha = \frac{1}{2} + \frac{N}{2\rho} < 1$ for $P = S_0$ or $\alpha = \max\{\frac{N}{2p}, \frac{1}{2} + \frac{N}{2\rho}\} < 1$ for $P = Q_0 + S_0$, and $\beta = 0$, we have that for any $1 < q \leq q_0$ and any perturbation as in (6.8) or (6.9) we have

$$P \in \mathcal{L}(X^\alpha, X^\beta)$$

and is a bounded family.

Then part i) follows by Theorem 3.13, Proposition 3.15. The analyticity of the semigroup follows from Theorem 6.12 while the order preserving from Proposition 5.3 in [5].

The estimates between uniform Lebesgue spaces follows from the sharp embeddings of the uniform Bessel spaces as in Lemma 6.13, some reiteration and observing that as soon as $q > N/2$ these spaces are included in $\dot{L}^\infty_U(I\!R^N) = BUC(I\!R^N)$.

The Hölder regularity of the solution follows by observing that the solution enters $\dot{H}^{2\gamma,q_0}_U(I\!R^N))$ for γ very close to 1 and $2\gamma - \frac{N}{q_0} > 0$ (or $2\gamma - \frac{N}{q_0} > 1$ if $q_0 > N$) and using the embeddings again.

Part ii) is consequence of Theorem 4.1 and the Hölder convergence follows as above.
■

Remark 6.15 *Note that if Theorem 6.12 was only proved for the Laplacian, then Theorem 3.20 would give Theorem 6.12 as stated.*

Remark 6.16 *After the theorem above we can apply the results in Theorem 3.7 for $f \in L^\sigma((0,T), \dot{H}^{2\gamma,q}_U(I\!R^N))$ with $1 \leq \sigma \leq \infty$, $1 < q \leq q_0$ and $\gamma \in [0,1)$.*

7 Elliptic regularity and convergence

Although the approach carried in this paper is of a "parabolic" nature rather than an "elliptic one", with the exception of Theorem 3.20, we show now that we can also derive some results on the undelying elliptic problems.

For this note that given the solutions operators $S_P(t)$, as in Theorem 3.13, and satisfying the estimates in Proposition 3.15, for any $\gamma \in E(\alpha) = (\alpha - 1, \alpha]$ (or any $\gamma \in I$ if $P = 0$), we can take any fix element $f \in X^\gamma$ and consider the corresponding solution of the nonhomogeneous problem

$$u(t; u_0) = S_P(t)u_0 + \int_0^t S_P(t-\tau) f \, d\tau \tag{7.1}$$

as in Theorem 3.7.

Assume moreover that $S_P(t)$ decays exponentially in the sense that

$$\|S_P(t)u_0\|_{\gamma'} \leq M_0 \mathrm{e}^{-\omega t} t^{-(\gamma' - \gamma)} \|u_0\|_\gamma, \quad \gamma' \geq \gamma. \tag{7.2}$$

for some $\omega > 0$. Note that we may let ω depend on γ, γ', but if the scale is nested we know from Lemma 3.2 that the exponent can be taken independent of the space in the scale.

Then we have

Proposition 7.1 *With the notations above*
i) The expression

$$\Phi = \int_0^\infty S_P(\tau) f \, d\tau$$

defines and element in $X^{\gamma'}$ *for any* $\gamma' < \gamma + 1$ *with*

$$\|\Phi\|_{\gamma'} \leq C\|f\|_\gamma$$

and such that for any $u_0 \in X^\gamma$ *we have the function in (7.1) satisfies* $u(t; u_0) \to \Phi$ *exponentially in* $X^{\gamma'}$ *as* $t \to \infty$.

If the semigroup $S_P(t)$ *is analytic, see Theorem 3.20, we have that* Φ *is the solution of the elliptic problem*

$$A\Phi = P\Phi + f.$$

ii) If $P^1, P^2 \in \mathcal{L}(X^\alpha, X^\beta)$, *with* $\|P^i\|_{\mathcal{L}(X^\alpha, X^\beta)} \leq R_0$ *are such that both semigroups* $S_{P^i}(t)$ *satisfy (7.2), and* $f_1, f_2 \in X^\gamma$, *we define*

$$\Phi_{P^i} = \int_0^\infty S_{P^i}(\tau) f_i \, d\tau, \quad i = 1, 2$$

and we have

$$\|\Phi_{P^1} - \Phi_{P^2}\|_{\gamma'} \leq C(\|f_1 - f_2\|_\gamma + \|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}\|f_2\|_\gamma).$$

Proof.
i) Observe that in (7.1) we have

$$\int_0^t S_P(t-\tau) f \, d\tau = \int_0^t S_P(\tau) f \, d\tau$$

and the exponential decay (7.2) gives the result.
ii) Now from Remark 4.2 the result is easy. ■

Remark 7.2 *Observe that if the scale is nested,* $S_{P^1}(t)$ *satisfies (7.2) and* $\|P^1 - P^2\|_{\mathcal{L}(X^\alpha, X^\beta)}$ *is sufficiently small then by Corollary 4.3,* $S_{P^2}(t)$ *also satisfies (7.2).*

We now particularize to the problems in Sections 5 and 6. Note that we have chose some particular but illustrative cases to show the scope of the results that Proposition 7.1 can give. First, using Theorem 5.6 and 5.9 we get

Theorem 7.3 *Assume that* m *is in* $L^p(\Omega)$, *with* $p > N/2$, m_0 *is in* $L^r(\Gamma)$ *and also* $\vec{d}$ *is in* $L^\rho(\Omega)^N$, *for* $\rho > N$. *Then, for any* $1 < q < \infty$, *and any* P *as in (5.10), (5.11) or (5.12) consider the semigroup,* $S_P(t)$ *in the space* $H^{2\gamma,q}_{bc}(\Omega)$ *for any* $\gamma \in I(q)$, *as in Theorem 5.6. Also denote*

$$s(q) = \sup I(q) = \begin{cases} 1 & \textit{if } P \textit{ equals } Q_0,\ S_0 \textit{ or } Q_0 + S_0 \\ 1 - \frac{1}{2q'} & \textit{if } P \textit{ equals } R_0,\ Q_0 + R_0,\ R_0 + S_0 \textit{ or } Q_0 + R_0 + S_0 \\ \frac{1}{2} & \textit{if } P \textit{ equals } T_0,\ Q_0 + T_0,\ R_0 + T_0 \textit{ or } Q_0 + R_0 + T_0. \end{cases}$$

Then there exists $\mu_0 = \mu_0(\|m\|_{L^p(\Omega)}, \|m_0\|_{L^r(\Gamma)}, \|\vec{d}\|_{L^\rho(\Omega)^N})$ *such that for all* $\mu > \mu_0$ *we have:*
i) For any $f \in L^q(\Omega)$, $1 < q < \infty$, *there exists a unique solution* Φ *of*

$$\int_\Omega a(x)\nabla\Phi\nabla\varphi + \int_\Omega (c(x)+\mu)\Phi\varphi + \int_{\partial\Omega} b(x)(x)\Phi\varphi =< P\Phi, \varphi > + \int_\Omega f\varphi$$

for all sufficiently smooth φ *and where in the definition of* P *one must take into account (5.1), (5.4) and (5.5). Moreover*

$$\Phi \in H^{2\gamma',q}_{bc}(\Omega), \quad \textit{for all } \gamma' < s(q), \quad \|\Phi\|_{H^{2\gamma',q}_{bc}(\Omega)} \leq C(\mu)\|f\|_{L^q(\Omega)}$$

and if $f \geq 0$ *then* $\Phi \geq 0$.
ii) For any $g \in L^s(\Gamma)$ *with* $1 \leq s \leq \infty$, *there exists* q_0 *such that for any* $1 < q < q_0$ *there exists some* $0 < \gamma_0' < \min\{\frac{1}{2} + \frac{1}{2q}, s(q)\}$ *such that there exists a unique solution* Φ *of*

$$\int_\Omega a(x)\nabla\Phi\nabla\varphi + \int_\Omega (c(x)+\mu)\Phi\varphi + \int_{\partial\Omega} b(x)(x)\Phi\varphi =< P\Phi, \varphi > + \int_\Gamma g\varphi$$

for all sufficiently smooth φ, *where in the definition of* P *one must take into account (5.1), (5.4) and (5.5), with*

$$\Phi \in H^{2\gamma',q}_{bc}(\Omega), \quad \textit{for all } \gamma' < \gamma_0', \quad \|\Phi\|_{H^{2\gamma',q}_{bc}(\Omega)} \leq C(\mu)\|g\|_{L^s(\Gamma)}$$

and if $g \geq 0$ *then* $\Phi \geq 0$.
iii) Finaly, if $m_\varepsilon \to m$ *in* $L^p(\Omega)$, $p > \frac{N}{2}$, $m_{0,\varepsilon} \to m_0$ *in* $L^r(\Gamma)$, $r > N-1$ *and* $\vec{d_\varepsilon} \to \vec{d}$ *in* $L^\rho(\Omega)^N$, $\rho > N$, *then the corresponding solutions* Φ_ε *satisfy*

$$\Phi_\varepsilon \to \Phi_0 \quad \textit{in} \quad H^{2\gamma',q}_{bc}(\Omega), \quad \textit{for all } \gamma' < \begin{cases} s(q) & \textit{in case i)} \\ \gamma_0' & \textit{in case ii).} \end{cases}$$

Proof. From Theorem 5.6, denoting μ_0 the exponent in (5.14) and taking the semigroup $S_{P-\mu I}(t)$, which decays exponentially for $\mu > \mu_0$ we can use Proposition 7.1.
i) In this case we have $\gamma = 0$ in Proposition 7.1 and from the expression of $I(q)$ in Theorem 5.6, we get the result since $s(q) < 1$ for all q.
ii) In this case take $1 < q < \infty$ and $\gamma > \frac{1}{2q'}$ such that $\gamma \geq \frac{1}{2} + \frac{N-1}{2s} - \frac{N}{2q}$. With this the embeddings of Bessel spaces in Section 2 imply that $L^s(\Gamma) \subset H^{-2\gamma,q}_{bc}(\Omega) = X^{-\gamma}$.

Now observe that

$$i(q) = \inf I(q) = \begin{cases} -1 & \text{if } P \text{ equals } Q_0,\, T_0 \text{ or } Q_0 + T_0 \\ -1 + \frac{1}{2q} & \text{if } P \text{ equals } R_0,\, Q_0 + R_0,\, R_0 + T_0 \text{ or } Q_0 + R_0 + T_0 \\ -\frac{1}{2} & \text{if } P \text{ equals } S_0,\, Q_0 + S_0,\, R_0 + S_0 \text{ or } Q_0 + R_0 + S_0\ . \end{cases}$$

and then we can take any γ such that

$$\gamma_0 := \max\{\frac{1}{2q'}, \frac{1}{2} + \frac{N-1}{2s} - \frac{N}{2q}\} < \gamma < -i(q).$$

Note that $\frac{1}{2q'} < -i(q)$ for all q and the second condition $\frac{1}{2} + \frac{N-1}{2s} - \frac{N}{2q} < -i(q)$ is satisfied for $1 < q < q_0$.

Then we get the result in Proposition 7.1, for any γ' such that

$$0 < 1 + i(q) < \gamma' < \min\{1 - \gamma_0, s(q)\} := \gamma'_0$$

and since $1 + i(q) < s(q)$ for all q this condition is nonvoid. Also, since $\gamma_0 > \frac{1}{2q'}$ then$\gamma'_0 < \frac{1}{2} + \frac{1}{2q}$.

In both cases the sign of Φ follows from the order preserving property of $S_P(t)$ and the expression for Φ in Proposition 7.1.

Part iii) follows from the secong part in Proposition 7.1. ■

Remark 7.4 *Note that q_0 in the Theorem above can be computed explicitly as follows:*

a) If $i(q) = -1$. Then if $s \geq N-1$ we have $q_0 = \infty$, while if $s < N-1$ we have $q_0 = \frac{Ns}{N-1-s}$.

b) If $i(q) = -1 + \frac{1}{2q}$. Then if $s \geq N-1$ we have $q_0 = \infty$, while if $s < N-1$ we have $q_0 = \frac{(N-1)s}{N-1-s}$.

c) If $i(q) = -\frac{1}{2}$. Then $q_0 = \frac{Ns}{N-1}$.

Also note that if $1 < q < \frac{Ns}{N-1}$ then $\gamma'_0 \geq \frac{1}{2}$.

Remark 7.5 *Note that the optimal value of the quantity μ_0 in the theorem is given by the principal eigenvalue of the following eigenvalue problem*

$$\int_\Omega a(x)\nabla u \nabla \varphi + \int_\Omega c(x) u\varphi + \int_{\partial\Omega} b(x)(x)\Phi\varphi =< Pu, \varphi > + \lambda \int_\Omega u\varphi$$

for all sufficiently smooth φ, which is characterized by the fact that it is the unique eigenvalue with a positive associated eigenfunction.

Now, for problems in $\mathbb{R}^N$, in an analogous way as before, from Theorems 6.5 and 6.7 and Proposition 7.1, we have

Theorem 7.6 *Assume that m is in $L^p_U(\mathbb{R}^N)$, with $p > N/2$, and $\vec{d}$ is in $L^\rho_U(\mathbb{R}^N)^N$, for $\rho > N$. For any $1 < q < \infty$, and any P as in (6.5), (6.6) consider the semigroup $S_P(t)$ in the space $H^{2\gamma,q}(\mathbb{R}^N)$ for any $\gamma \in I(q)$ as in Theorem 6.5. Also denote*

$$s(q) = \sup I(q) = \begin{cases} 1 & \text{if } P \text{ equals } Q_0,\ S_0 \text{ or } Q_0 + S_0 \\ \frac{1}{2} & \text{if } P \text{ equals } T_0,\ Q_0 + T_0. \end{cases}$$

Then there exists $\mu_0 = \mu_0(\|m\|_{L^p_U(\mathbb{R}^N)}, \|\vec{d}\|_{L^\rho_U(\mathbb{R}^N)^N})$ such that for all $\mu > \mu_0$ we have: i) For any $f \in L^q(\mathbb{R}^N)$, $1 < q < \infty$, there exists a unique solution Φ of

$$\int_{\mathbb{R}^N} \nabla\Phi\nabla\varphi + \mu \int_{\mathbb{R}^N} \Phi\varphi =< P\Phi, \varphi > + \int_{\mathbb{R}^N} f\varphi$$

for all sufficiently smooth φ, and

$$\Phi \in H^{2\gamma',q}(\mathbb{R}^N), \quad \text{for all } \gamma' < s(q), \quad \|\Phi\|_{H^{2\gamma',q}(\mathbb{R}^N)} \leq C(\mu)\|f\|_{L^q(\mathbb{R}^N)}$$

and if $f \geq 0$ then $\Phi \geq 0$.
ii) If $m_\varepsilon \to m$ in $L^p_U(\mathbb{R}^N)$, $p > \frac{N}{2}$ and $\vec{d}_\varepsilon \to \vec{d}$ in $L^\rho_U(\mathbb{R}^N)^N$, $\rho > N$ then the corresponding solutions Φ_ε satisfy

$$\Phi_\varepsilon \to \Phi_0 \quad \text{in} \quad H^{2\gamma',q}(\mathbb{R}^N), \quad \text{for all } \gamma' < s(q).$$

Finally, for problems in uniform spaces, from Theorem 6.14 we get

Theorem 7.7 *Assume that m is in $\dot{L}^p_U(\mathbb{R}^N)$, with $p > N/2$, and $\vec{d}$ is in $\dot{L}^\rho_U(\mathbb{R}^N)^N$, for $\rho > N$ and define $q_0 = \min\{p, \rho\} > N/2$.*

Then there exists $\mu_0 = \mu_0(\|m\|_{\dot{L}^p_U(\mathbb{R}^N)}, \|\vec{d}\|_{\dot{L}^\rho_U(\mathbb{R}^N)^N})$ such that for all $\mu > \mu_0$ we have:
i) For any $f \in \dot{L}^q_U(\mathbb{R}^N)$, $1 < q \leq q_0$, there exists a unique solution Φ of

$$-\Delta\Phi + \mu\Phi = P\Phi + f$$

and

$$\Phi \in \dot{H}^{2\gamma',q}_U(\mathbb{R}^N), \quad \text{for all } \gamma' < 1, \quad \|\Phi\|_{\dot{H}^{2\gamma',q}_U(\mathbb{R}^N)} \leq C(\mu)\|f\|_{\dot{L}^q_U(\mathbb{R}^N)}$$

and if $f \geq 0$ then $\Phi \geq 0$.
ii) If $m_\varepsilon \to m$ in $\dot{L}^p_U(\mathbb{R}^N)$, $p > \frac{N}{2}$ and $\vec{d}_\varepsilon \to \vec{d}$ in $\dot{L}^\rho_U(\mathbb{R}^N)^N$, $\rho > N$ then the corresponding solutions Φ_ε satisfy

$$\Phi_\varepsilon \to \Phi_0 \quad \text{in} \quad \dot{H}^{2\gamma',q}_U(\mathbb{R}^N), \quad \text{for all } \gamma' < 1.$$

8 Final remarks

It should be noted that on most of the results in Sections 3 and 4 the fact that the semigroup is strongly continuous, that is continuous at $t = 0$, is used in a very few steps.

For example, it is first used in the proof of Lemma 3.10 and then in Lemma 3.11. These results are later used in Proposition 3.12 and in Theorem 3.13. Continuity at $t = 0$ is finally used in the last part of the proof of Lemma 3.22.

Certainly in many applications the underlying semigroups are strongly continuous, as we saw on Sections 2, 5 or 6 although there are other important case in which it is not. For example in Section 6 we found that the heat equation generates non strongly continuous, analytic semigroups in uniform spaces. The same situation occurs if one takes the heat equation in $L^\infty(\Omega)$ due to the smoothing effect on the solutions. In all these cases the semigroup is strongly continuous for large (but not dense) classes of initial data, namely on "dotted" uniform spaces of $BUC(\overline{\Omega})$ respectively.

The results in Section 3 and 4 mentioned above could be adapted to that situation by assuming some restrictions on the initial data like in the examples we just mentioned.

In this way the first part of Lemma 6.8 can be used to prove a similar result to Theorem 6.9 in undotted space, which would give analogous results to Proposition 3.1 and 3.2 in [5].

Also Proposition 5.1 in [5] is the analogous result to Theorem 6.12 in undotted spaces and could be used to obtain an analogous result to Theorem 6.14 in those spaces. As there would be some minor differences in the statements, due to subtle technical conditions we have not pursued this line on the main part of this paper.

On the other hand, note that also the fact that the original semigroup $S(t)$ is analytic has been used scarcely. Indeed it is used for the first time in Remark 3.18. It is later used in Theorem 3.20 and in the proof of part iii) of Lemma 3.22. Certainly the semigroups in Sections 2, 5 and 6 are analytic but most of the results in Sections 3 and 4 go along without this assumption.

Another situation that can be handled with the techniques in Sections 3 and 4 is the case of semigroups with "defects". By this we mean semigroups that instead of satisfying (3.3) have a defect of the type

$$\|S(t)\|_{\beta,\alpha} := \|S(t)\|_{\mathcal{L}(X^{\beta},X^{\alpha})} \leq \frac{M_0(\beta,\alpha,T)}{t^{\alpha-\beta+\mu}}, \qquad \text{for all} \quad 0 < t \leq T$$

for some constant $M_0(\beta,\alpha,T) > 0$ and $0 < \mu < 1$. Note that in particular, the semigroup is singular at $t = 0$ even for $\beta = \alpha$. One can find this type of semigroups with deffects, for example, in parabolic problems in "dumbbell" domains, see [4]; see also [13]. Hence, as in theses references, we assume that, for each α, the semigroup is continuous for $t > 0$ and continuous at $t = 0$ for a dense set of intial data in X^{α}.

In such a case, note that the first part of Theorem 3.7 holds under the assumption that

$$0 \leq \gamma' - \gamma < \frac{1}{\sigma'} - \mu,$$

and $u \in \mathcal{L}^{\infty}_{\gamma'-\gamma+\mu}((0,T), X^{\gamma'})$.

In Lemma 3.9, (3.11) must be replaced by

$$\beta \leq \gamma' < \beta + 1 - \mu \quad \text{and} \quad 0 \leq \varepsilon < 1$$

and the right hand side in parts i) and ii) have a term $t^{\beta+\delta+1-\gamma'-\varepsilon-\mu}$. Also, (3.12) now reads

$$\delta = \gamma' - \gamma + \mu \geq 0 \quad \text{and} \quad \gamma < \beta + 1 - \varepsilon.$$

Note that the proof of Lemma 3.10 breaks down in several places, while in Lemma 3.11 we require now

$$\beta \leq \gamma' < \beta + 1 - \varepsilon - \mu, \quad 0 \leq \varepsilon < 1.$$

and we do not prove now continuity at $t = 0$ of the integral term in (3.10), (which would require the stronger conditon $\varepsilon < 1 - \mu$) but we prove that $\mathcal{F}(u, u_0)(t)$ behaves, as $t \to 0$ as $t^{-\mu}$.

In Proposition 3.12 we replace (3.14) by

$$0 \leq \alpha - \beta < 1 - \mu$$

and prove that (3.10) is a contraction in $\mathcal{L}^{\infty}_{\mu}((0,T],X^{\alpha})$. In the proof of the Proposition we don't use now Lemma 3.10 and use Lemma 3.11 with $\gamma' = \alpha$ and $\varepsilon = \mu$.

Then in Theorem 3.13 we use again $0 \leq \alpha - \beta < 1 - \mu$ while (3.16) and (3.17) are replaced respectively by

$$\alpha - 1 + \mu < \gamma \leq \alpha,$$

and

$$\beta \leq \gamma' < \beta + 1 - \mu, \quad \gamma' \geq \gamma.$$

In the proof of the theorem we prove that (3.10) is a contraction in $\mathcal{L}^{\infty}_{\varepsilon}((0,T],X^{\alpha})$ with $\varepsilon = \alpha - \gamma + \mu$ and we don't use now Lemma 3.10. Note that the ranges above define the new sets

$$E(\alpha) = (\alpha - 1 + \mu, \alpha], \quad R(\beta) = [\beta, \beta + 1 - \mu)$$

and solutions (3.18) define the semigroup $S_P(t)$ for $\gamma \in [\beta, \alpha]$ as before.

Finally Proposition 3.15 holds with the sets $E(\alpha)$ and $R(\beta)$ just mentioned and in (3.20) we have now

$$\|S_P(t)u_0\|_{\gamma'} \leq M_0 \mathrm{e}^{\omega t} t^{-(\gamma'-\gamma)-\mu} \|u_0\|_{\gamma}, \quad \gamma' \geq \gamma.$$

Finally note that the abstract results in Sections 3 and 4 are potentially applicable to many different perturbation problems. We have focused here, in Section 2, 5 and 6, into some particular examples of practical interest in many applications of partial differential equations, although the application of the theoretical results are not limited to such examples.

References

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, Boston, 1978.

[2] H. Amann. Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In *Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992)*, volume 133 of *Teubner-Texte Math.*, pages 9–126. Teubner, Stuttgart, 1993.

[3] H. Amann. *Linear and quasilinear parabolic problems. Vol. I*, volume 89 of *Monographs in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1995. Abstract linear theory.

[4] J. M. Arrieta, A. N. Carvalho, and G. Lozada-Cruz. Dynamics in dumbbell domains. II. The limiting problem. *J. Differential Equations*, 247(1):174–202, 2009.

[5] J.M. Arrieta, J. W. Cholewa, T. Dlotko, and A. Rodríguez-Bernal. Linear parabolic equations in locally uniform spaces. *Math. Models Methods Appl. Sci.*, 14(2):253–293, 2004.

[6] J.M. Arrieta, J.W. Cholewa, T. Dlotko, and A. Rodríguez-Bernal. Asymptotic behavior and attractors for reaction diffusion equations in unbounded domains. *Nonlinear Anal.*, 56(4):515–554, 2004.

[7] J. Bergh and J. Löfström. *Interpolation spaces. An introduction.* Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[8] T. Cazenave and A. Haraux. *Introduction aux problèmes d'évolution semi-linéaires.* Ellipses, 1990.

[9] D. Daners. Heat kernel estimates for operators with boundary conditions. *Math. Nachr.*, 217:13–41, 2000.

[10] A. Friedman. *Partial differential equations.* Holt, Rienehart, Winston, 1969.

[11] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1981.

[12] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems.* Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995.

[13] N. Okazawa. A generation theorem for semigroups of growth order α. *Tôhoku Math. J. (2)*, 26:39–51, 1974.

[14] H. Triebel. *Interpolation theory, function spaces, differential operators.* Johann Ambrosius Barth, Heidelberg, second edition, 1995.

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