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LINEAR AND SEMILINEAR HIGHER ORDER PARABOLIC EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this paper we consider some fourth order linear and semilinear equations in $\mathbb{R}^N$ and make a detailed study of the solvability of the Cauchy problem. For the linear equation we consider some weakly integrable potential terms, and for any $1 < p < \infty$ prove that for a suitable family of Bessel potential spaces, $H^{\alpha}_p(\mathbb{R}^N)$, the linear equation defines a strongly continuous analytic semigroup.

Using this result, we prove that the nonlinear problems we consider can be solved for initial data in $L^p(\mathbb{R}^N)$ and in $H^2_p(\mathbb{R}^N)$. We also find the corresponding critical exponents, that is, the largest growth allowed for the nonlinear terms for these classes of initial data.

1. Introduction

In this article we consider the following Cauchy problem in $\mathbb{R}^N$,

$$\begin{cases} u_t + \Delta^2 u = f(x,u), \quad t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x),\ x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the nonlinear term is assumed to be of the general form

$$f(x,u) = g(x) + m(x)u + f_0(x,u),\ x \in \mathbb{R}^N,\ u \in \mathbb{R}, \tag{1.2}$$

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for some suitable m, g described below and

$$f_0 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz in } u \in \mathbb{R} \text{ uniformly for } x \in \mathbb{R}^N, \tag{1.3}$$

and

$$f_0(x,0) = 0, \quad \frac{\partial f_0}{\partial u}(x,0) = 0, \; x \in \mathbb{R}^N. \tag{1.4}$$

In some cases, depending on the space in which we solve (1.1), we will also require a growth condition in f_0 of the form

$$|f_0(x,u_1) - f_0(x,u_2)| \le c|u_1 - u_2|(1 + |u_1|^{\rho-1} + |u_2|^{\rho-1}), \; u_1, u_2 \in \mathbb{R}. \tag{1.5}$$

for some $\rho > 1$ and $c > 0$.

Hence, our goal is to give suitable conditions on g, m and ρ under which (1.1) has a local solution for certain classes of initial data. Here we consider initial data in some space of Bessel potentials, which we generically denote $H^\alpha_p(\mathbb{R}^N)$, (see [14]). When $p = 2$ we will denote these spaces as $H^\alpha(\mathbb{R}^N)$ which are Hilbert spaces. In particular, we are interested in the cases when $u_0 \in L^p(\mathbb{R}^N)$ and $u_0 \in H^2_p(\mathbb{R}^N)$.

In order to obtain local solutions with low regularity conditions on m we must first study in detail the solutions of linear equations of the form

$$\begin{cases} u_t + \Delta^2 u = C(x)u, & t > 0, \; x \in \mathbb{R}^N, \\ u(0) = u_0 \end{cases} \tag{1.6}$$

with initial data in Bessel spaces $H^\alpha_p(\mathbb{R}^N)$. Here, we will assume that

$$C \in L^r_U(\mathbb{R}^N), \quad \max\{\frac{N}{4}, 1\} < r \le \infty, \tag{1.7}$$

where this space is defined, for $1 \le r \le \infty$ as

$$L^r_U(\mathbb{R}^N) \stackrel{def}{=} \{\phi \in L^r_{loc}(\mathbb{R}^N) : \|\phi\|_{L^r_U(\mathbb{R}^N)} = \sup_{y \in \mathbb{R}^N} \|\phi\|_{L^r(B(y,1))} < \infty\}$$

(see [5, 10] and note that $L^\infty_U(\mathbb{R}^N) := L^\infty(\mathbb{R}^N)$).

First, regarding the linear problem (1.6) we will prove, among other, the following result.

Theorem 1.1. *Suppose that $C \in L^r_U(\mathbb{R}^N)$ and $r > \max\{\frac{N}{4}, 1\}$.*
i) Then the operator $A_C = \Delta^2 - C(x)I$ is a sectorial operator in $L^p(\mathbb{R}^N)$ and $-A_C$ generates a C^0 analytic semigroup, $\{e^{-A_C t} : t \ge 0\}$, in $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.
ii) The scale of fractional power spaces, $\{E^\alpha_p, \; \alpha \in \mathbb{R}\}$, associated to this operator, is given by

$$E^\alpha_p = \begin{cases} H^{4\alpha}_p(\mathbb{R}^N) & \text{for } 0 \le \alpha \le \beta^*(p) \le 1, \\ (H^{-4\alpha}_{p'}(\mathbb{R}^N))' & \text{for } -1 \le -\beta_*(p) \le \alpha < 0, \end{cases} \tag{1.8}$$

with $0 < \beta^*(p) = 1 + \left(\frac{N}{4p} - \frac{N}{4r}\right)_- \le 1$ and $\beta_*(p) = \beta^*(p') = 1 + \left(\frac{N}{4p'} - \frac{N}{4r}\right)_-$, where $x_- = \min\{x, 0\}$ denotes the negative part of $x \in \mathbb{R}$.

iii) On this scale of spaces, the analytic semigroup generated by $-A_C$ satisfies, for some $\omega \in \mathbb{R}$,

$$\|e^{-A_C t}\|_{\mathcal{L}(E_p^\sigma, E_p^\xi)} \leq M \frac{e^{-\omega t}}{t^{\xi-\sigma}} \quad t > 0, \ -\beta_*(p) \leq \sigma \leq \xi \leq \beta^*(p). \tag{1.9}$$

iv) Also, if $p = 2$ then (1.9) is satisfied for some $\omega > 0$ if and only if there is a certain $\omega_0 > 0$ such that

$$\int_{\mathbb{R}^N} (|\Delta\phi|^2 - C(x)\phi^2) \geq \omega_0 \|\phi\|^2_{L^2(\mathbb{R}^N)}, \tag{1.10}$$

for all $\phi \in H^2(\mathbb{R}^N)$. We say then that the C^0 analytic semigroup $\{e^{-A_C t} : t \geq 0\}$ in $L^2(\mathbb{R}^N)$ is exponentially decaying as $t \to \infty$.

Remark 1.2. *i) Observe that $\beta^*(p) = 1$ iff $r \geq p$ and, for all $1 < p < \infty$,*

$$\beta^*(p) \geq 1 - \frac{N}{4r} > 0.$$

Hence, the interval $[-\beta_*(p), \beta^*(p)]$ contains at least the symmetric interval

$$[-1 + \frac{N}{4r}, 1 - \frac{N}{4r}].$$

Also, the length of the interval $[-\beta_*(p), \beta^*(p)]$ is $L = \beta^*(p) + \beta^*(p')$ and then

$$L = \begin{cases} 1 + \beta^*(p'), & \text{if } p' \geq r \geq p \\ 1 + \beta^*(p), & \text{if } p \geq r \geq p' \\ 2 & \text{if } r \geq p, p' \\ 2 + \frac{N}{4} - \frac{N}{2r} & \text{if } p, p' \geq r. \end{cases}$$

Note that in any case $L > 1$ since $r > \frac{N}{4}$ and $r > 1$.
ii) Note that we can use the usual notation

$$H_p^{-4\alpha}(\mathbb{R}^N) = (H_{p'}^{-4\alpha}(\mathbb{R}^N))' \quad \alpha > 0$$

and then (1.8) becomes $E_p^\alpha = H_p^{4\alpha}(\mathbb{R}^N)$ for $\alpha \in [-\beta_*(p), \beta^*(p)]$.
iii) Note that it is implicit in (1.10) that since C satisfies (1.7) and $\phi \in H^2(\mathbb{R}^N)$, then $C\phi^2 \in L^1(\mathbb{R}^N)$, see (2.24).

It is worth stressing that in Theorem 1.1 there is no restriction other than $r > \frac{N}{4}$ and $1 < p < \infty$. However the Theorem reflects that, depending on the comparison of r with p or p', the range of spaces in (1.8) for which we have a nice semigroup as in (1.9) is biased to either negative or positive indexes. In fact the case $r \geq p$ (and hence $\beta^*(p) = 1$) reflects that the potential is suitable integrable with respect to the base space, $L^p(\mathbb{R}^N)$. Hence, in this case the potential can be naturally handled as a perturbation of the bi-Laplacian operator. In particular $C\phi \in L^p(\mathbb{R}^N)$ for any function $\phi \in D(\Delta^2) = H_p^4(\mathbb{R}^N)$. See for example [2] for a similar situation for the case of second order operators.

On the other hand when $r < p$, the potential is poorly integrable with respect to the base space and it is more difficult to handle as a perturbation of the bi-Laplacian. In particular $C\phi$ is not in $L^p(\mathbb{R}^N)$ for every function in $H_p^4(\mathbb{R}^N)$. Hence, the potential must be treated as a perturbation of Δ^2 in a weaker space than $L^p(\mathbb{R}^N)$; see the comment after Lemma 2.3. For this we make use of the extrapolated scale of fractional power spaces of Δ^2, for which we follow the general construction in [1]. These spaces are precisely the Bessel spaces $H^{\alpha}_p(\mathbb{R}^N)$ with $\alpha \in [-1, 1]$ and therefore Theorem 1.1 also states that in a portion of this scale the operator $A_C = \Delta^2 - C(x)I$ is a nicely behaved operator.

In order to make precise the remaining assumptions on (1.1), we will assume that in (1.2) we have

$$m \in L^r_U(\mathbb{R}^N), \quad \max\{\frac{N}{4}, 1\} < r \leq \infty \tag{1.11}$$

and, for simplicity,

$$g \in L^p(\mathbb{R}^N), \quad 1 < p < \infty. \tag{1.12}$$

Then we have the following results on the local existence of (1.1).

Theorem 1.3. *Let $1 < p < \infty$, assume (1.2)–(1.4), (1.11), (1.12) and suppose that (1.5) holds with some*

$$1 < \rho \leq \rho^1_c := 1 + \frac{4p}{N}$$

Then (1.1) is locally well posed in $L^p(\mathbb{R}^N)$.

Now we consider local well posedness in $H^2_p(\mathbb{R}^N)$, $1 < p < \infty$. For this, note that we need that the scale of spaces in (1.8) contains $H^2_p(\mathbb{R}^N)$, which requires

$$\beta^*(p) = 1 + \Big(\frac{N}{4p} - \frac{N}{4r}\Big)_- > \frac{1}{2}, \tag{1.13}$$

that is, $\frac{N}{r} - \frac{N}{p} < 2$.

Note that (1.13) is satisfied if $r \geq p$ or if $p \leq \frac{N}{2}$, since $r > \frac{N}{4}$. Also, (1.13) is satisfied for $p = 2$ since $r > \max\{\frac{N}{4}, 1\}$. Hence, we have

Theorem 1.4. *Assume (1.2)–(1.4), (1.11) and (1.12). Then the problem (1.1) is locally well posed in $H^2_p(\mathbb{R}^N)$, with $1 < p < \infty$, provided that (1.12) holds and either*

(i) $2 > \frac{N}{p} > \frac{N}{r} - 2$
(ii) $2 = \frac{N}{p}$ *and (1.5) holds with some* $1 < \rho < \infty$,
(iii) $2 < \frac{N}{p}$ *and (1.5) holds with some* $1 < \rho \leq \rho^2_c := 1 + \frac{4p}{N-2p}$.

In both Theorems 1.3 and 1.4 a solution of (1.1) with an initial value $u_0 \in L^p(\mathbb{R}^N)$, or $u_0 \in H^2_p(\mathbb{R}^N)$ respectively, is defined on the maximal interval of existence $[0, \tau_{u_0})$ and satisfies on this interval the variation of constants formula

$$u(t) = e^{(-\Delta^2+mI)t}u_0 + \int_0^t e^{(-\Delta^2+mI)(t-s)}\big(f_0(\cdot, u(s)) + g\big)ds, \tag{1.14}$$

where $e^{(-\Delta^2+mI)t}$ is the semigroup in $L^p(\mathbb{R}^N)$ as in Theorem 1.1. Furthermore, if $u_0 \in L^p(\mathbb{R}^N)$ then

$$u \in C([0, \tau_{u_0}), L^p(\mathbb{R}^N)) \cap C((0, \tau_{u_0}), H^{4\beta^*(p)}_p(\mathbb{R}^N)) \cap C^1((0, \tau_{u_0}), L^p(\mathbb{R}^N)), \tag{1.15}$$

while if $u_0 \in H^2_p(\mathbb{R}^N)$

$$u \in C([0, \tau_{u_0}), H^2_p(\mathbb{R}^N)) \cap C((0, \tau_{u_0}), H^{4\beta^*(p)}_p(\mathbb{R}^N)) \cap C^1((0, \tau_{u_0}), L^p(\mathbb{R}^N)), \tag{1.16}$$

where $\beta^*(p) = 1 + \left(\frac{N}{4p} - \frac{N}{4r}\right)_-$ as in Theorem 1.1.

Concerning the maximal interval of existence of a solution note that in the "subcritical" cases, that is when $\rho < \rho_c^1$, or $\rho < \rho_c^2$, respectively, it has the property that

$$\tau_{u_0} < \infty \quad \text{implies} \quad \limsup_{t \to \tau_{u_0}^-} \|u(t)\|_{L^p(\mathbb{R}^N)} = \infty, \tag{1.17}$$

or respectively,

$$\limsup_{t \to \tau_{u_0}^-} \|u(t)\|_{H^2_p(\mathbb{R}^N)} = \infty \tag{1.18}$$

(see [9, Theorem 3.3.4] and [6, Corollary 1.1]). The critical cases, that is when $\rho = \rho_c^1$, or $\rho = \rho_c^2$, respectively are more involved and (1.17), (1.18) are not true in general; see [6] for related results.

Note that for solving the nonlinear problem (1.1) we use the technique developed in [3], which requires checking some properties of the Nemitsky nonlinear operator associated to the nonlinear term in (1.1), between suitable spaces of the ones appearing in Theorem 1.1. Also, with this approach, (1.1) can be solved for initial data in other spaces of Bessel potentials. Nonetheless we intentionally focus here on the spaces $L^p(\mathbb{R}^N)$ and $H^2_p(\mathbb{R}^N)$ as they appear naturally when studying asymptotic behavior of the solutions, see [8].

Also, in Theorems 1.3 and 1.4 the assumption on g can be suitably weakened. For example we can allow g to belong to some $H^s_p(\mathbb{R}^N)$ spaces for $s < 0$. This however would make the solutions to be less regular than in (1.15) or (1.16). As we have focused in solving (1.1) with linear potential as in (1.11), we have not pursued this weaker regularity of g in this paper.

Finally, it is worth noting that although the range of suitable spaces for the linear equation changes with r and p, as seen in Theorem 1.1, this has no effect in existence results in Theorems 1.3 and 1.4 nor in the the critical exponents appearing in these results.

Therefore, in Section 2 we analyze in detail the solutions of the linear equation (1.6) with potentials as in (1.7). In particular, we prove Theorem 1.1. On the other hand, in Section 3 we will prove Theorems 1.3 and 1.4.

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2. On some fourth order linear parabolic equations in $L^p(\mathbb{R}^N)$

We first prove that the bi-Laplacian operator Δ^2 in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, is a sectorial operator as in [9, Definition 1.3.1].

Proposition 2.1. *For any $1 < p < \infty$ the bi-Laplacian operator*

$$A = \Delta^2$$

considered in $L^p(\mathbb{R}^N)$ with domain $H^4_p(\mathbb{R}^N)$ is a densely defined sectorial operator. Consequently, $-A$ generates in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, a C^0 analytic semigroup $\{e^{-\Delta^2 t}\}$.

Proof: Recall from [9, §1.6] that the resolvent of the Laplacian Δ in $L^p(\mathbb{R}^N)$, $1 < p < \infty$, satisfies the estimate

$$\|(\lambda I - \Delta)^{-1}\phi\|_{L^p(\mathbb{R}^N)} \leq \frac{c}{(\cos\frac{\theta}{2})^{\frac{N}{2}+1}|\lambda|}\|\phi\|_{L^p(\mathbb{R}^N)} \tag{2.1}$$

for any $|arg\lambda| \leq \theta < \pi$. Therefore, whenever $0 < \beta \leq |arg\lambda| \leq \pi$, the equation

$$(\lambda I - \Delta^2)\psi = \phi \in L^p(\mathbb{R}^N)$$

has the unique solution $\psi \in H^4_p(\mathbb{R}^N)$ given by

$$\psi = -(\omega I - \Delta)^{-1}(-\omega I - \Delta)^{-1}\phi$$

with ω such that $\omega^2 = \lambda$ and thus satisfying $\frac{\beta}{2} \leq |arg(\pm\omega)| \leq \pi - \frac{\beta}{2}$. From (2.1) we get

$$\|(\lambda I - \Delta^2)^{-1}\phi\|_{L^p(\mathbb{R}^N)} \leq \frac{c^2}{(\cos\frac{2\pi-\beta}{4})^{N+2}|\lambda|}\|\phi\|_{L^p(\mathbb{R}^N)},\ \phi \in L^p(\mathbb{R}^N)$$

whenever $\beta \leq |arg\lambda| \leq \pi$ and $\beta \in \left(0, \frac{\pi}{2}\right)$, which proves the result. □

Denote by X^α_p, $\alpha \geq 0$, the fractional power spaces associated with A in $L^p(\mathbb{R}^N)$, $p > 1$. Recall from [9, p. 29] that these spaces are defined as the domains of fractional powers of $A + \lambda I$, where λ is any constant such that $Re\, \sigma(A + \lambda I) > 0$; thus in what follows we set

$$X^\alpha_p = D((A + I)^\alpha),\ \alpha \geq 0,\ 1 < p < \infty.$$

With the aid of complex interpolation method the spaces X^α_p can be characterized in terms of the spaces of Bessel potentials. This is immediate when $p = 2$ as in this case A is a selfadjoint operator in $L^2(\mathbb{R}^N)$ and the imaginary powers $(A + I)^{it}$ are unitary operators in $L^2(\mathbb{R}^N)$ (see [14, 1.18.10]). For $p = 2$ we thus have the characterization

$$X^\alpha_p = [L^2(\Omega), H^4_p(\mathbb{R}^N)]_\alpha = H^{4\alpha}_p(\mathbb{R}^N),\ \alpha \in (0, 1). \tag{2.2}$$

In what follows we prove that (2.2) holds for any $1 < p < \infty$.

Proposition 2.2. *For every $1 < p < \infty$ and $\alpha \in (0, 1)$ the fractional power space X^α_p associated with $A = \Delta^2$ in $L^p(\mathbb{R}^N)$ coincides with the Bessel potentials space $H^{4\alpha}_p(\mathbb{R}^N)$.*

Proof: Let $\Lambda = -\Delta$ in $L^p(\mathbb{R}^N)$, $p > 1$, and note that [11, Theorem 10.6] yields:

$$[(\Lambda + I)^2]^\alpha = (\Lambda + I)^{2\alpha},\ \alpha > 0; \tag{2.3}$$

in particular

$$[(\Lambda + I)^2]^{\frac{1}{2}} = (\Lambda + I).$$

By interpolation (see [14]) we have

$$\|\Delta\phi\|_{L^p(\mathbb{R}^N)} \leq \|\phi\|_{H^2_p(\mathbb{R}^N)} \leq c\|\phi\|^{\frac{1}{2}}_{H^4_p(\mathbb{R}^N)}\|\phi\|^{\frac{1}{2}}_{L^p(\mathbb{R}^N)},\ \phi \in H^4_p(\mathbb{R}^N).$$

Recalling that the norms $\|(\Delta^2 + I)\phi\|_{L^p(\mathbb{R}^N)}$ and $\|\phi\|_{H^4_p(\mathbb{R}^N)}$ are equivalent (see [14, §2.5.3 Step 3]) we infer that 2Δ is a relatively bounded perturbation of $\Delta^2 + I$ and $\Delta^2 + I + 2\Delta = (\Lambda + I)^2$ is sectorial in $L^p(\mathbb{R}^N)$. Since $(\Lambda + I)^2 - (A + I) = -2\Delta$ and $\Delta[(\Lambda + I)^2]^{-\frac{1}{2}} = \Delta(\Lambda + I)^{-1} = -I + (\Lambda + I)^{-1}$ is a bounded operator in $L^p(\mathbb{R}^N)$ the domains of $[(\Lambda + I)^2]^\alpha$

coincide for $\alpha \in (0,1)$ with the domains of $(A+I)^\alpha$ (see [9, Theorem 1.4.8]). Combining this with (2.3) we get

$$X_p^\alpha = D((A+I)^\alpha) = D((\Lambda+I)^{2\alpha}).$$

Since the domains of $(\Lambda+I)^s$, $s>0$, are known to coincide with $H_p^{2s}(\mathbb{R}^N)$ (see [7, (1.3.62)]), the proof is complete. □

Following [1, p. 262] we can now construct the extrapolated fractional power scale for $A=\Delta^2$. For this denote by $(X_p)^{-1}$ the completion of $X_p = L^p(\mathbb{R}^N)$ under the $\|(A+I)^{-1}\cdot\|_{X_p}$-norm. Then $A+I$ extends uniquely to sectorial operator $A+I : X_p \to (X_p)^{-1}$. Consider then the one-sided fractional power scale

$$\hat{X}_p^\alpha := \left((X_p)^{-1}\right)^\alpha,\ \alpha \geq 0. \tag{2.4}$$

Note that, due to Proposition 2.2 and [1, Theorem V.1.4.12],

$$\hat{X}_p^\alpha = \begin{cases} H_p^{4(\alpha-1)}(\mathbb{R}^N),\ \alpha \in [1,2], \\ (H_{p'}^{4(1-\alpha)}(\mathbb{R}^N))',\ \alpha \in [0,1), \end{cases} \tag{2.5}$$

where one can also use the usual notation letting

$$H_p^{-s}(\mathbb{R}^N) = (H_{p'}^s(\mathbb{R}^N))' \ \text{ for } \ s>0.$$

The above results corresponds to $C=0$ in Theorem 1.1 whereas our further concern is the situation when C is a non-zero potential as in (1.7). In what follows we thus consider a multiplication operator Q_C defined by $C:\mathbb{R}^N \to \mathbb{R}$; namely

$$Q_C(\phi)(x) = C(x)\phi(x),\ x \in \mathbb{R}^N,$$

for any function $\phi : \mathbb{R}^N \to \mathbb{R}$. Our concern will be to describe a portion of the scale in which Q_C is continuous. Note that for a poorly integrable potential any 'target space' of Q_C is rather expected to belong to a negative portion of the scale.

We will consider potentials of the class $L_U^r(\mathbb{R}^N)$, as in (1.7), and whenever $s \in (1,\infty)$ we will write

$$\beta^*(s) := 1 + \left(\frac{N}{4s} - \frac{N}{4r}\right)_- .$$

With this set-up we will prove a technical lemma, which builds upon embedding properties of spaces of Bessel potentials.

Lemma 2.3. *Suppose that $C \in L_U^r(\mathbb{R}^N)$ with $r > \max\{\frac{N}{4},1\}$, $p \in (1,\infty)$ and let β be any number from the interval*

$$I(p) = (-\beta^*(p'), \beta^*(p)-1] \subset (-1,0].$$

Then, there is a certain interval $(\alpha_0, 1+\beta)$ such that for any $\alpha \in (\alpha_0, 1+\beta)$ we have an estimate of the form

$$\|C\phi\|_{(H_{p'}^{-4\beta}(\mathbb{R}^N))'} \leq c\|C\|_{L_U^r(\mathbb{R}^N)}\|\phi\|_{H_p^{4\alpha}(\mathbb{R}^N)};$$

equivalently

$$Q_C \in \mathcal{L}(H_p^{4\alpha}(\mathbb{R}^N), (H_{p'}^{-4\beta}(\mathbb{R}^N))') \ \ \textit{and} \ \ \|Q_C\|_{\mathcal{L}(H_p^{4\alpha}(\mathbb{R}^N),(H_{p'}^{-4\beta}(\mathbb{R}^N))')} \leq c\|C\|_{L_U^r(\mathbb{R}^N)}.$$

Proof: The main idea that drives the argument can be found in [13, Lemma 6.4 part i)].

We observe that

$$\|C\phi\|_{(H^{-4\beta}_{p'}(\mathbb{R}^N))'} = \sup_{\|\psi\|_{H^{-4\beta}_{p'}(\mathbb{R}^N)}=1} \Big|\int_{\mathbb{R}^N} C\phi\psi\Big|$$

and we cover $\mathbb{R}^N$ with cubes Q_i, $i \in \mathbb{Z}^N$, centered at $i \in \mathbb{Z}^N$ and having unitary edges parallel to the axes so that $\mathbb{R}^N = \cup_{i\in\mathbb{Z}^N}\overline{Q_i}$, where $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Hence, we get

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} C\phi\psi\Big| &\leq \sum_{i\in\mathbb{Z}^N}\int_{Q_i} |C||\phi||\psi| \leq \sum_{i\in\mathbb{Z}^N} \|C\|_{L^{p_1}(Q_i)}\|\phi\|_{L^{p_2}(Q_i)}\|\psi\|_{L^{p_3}(Q_i)} \\ &\leq \|C\|_{L^r_U(\mathbb{R}^N)} \sum_{i\in\mathbb{Z}^N} \|\phi\|_{L^{p_2}(Q_i)}\|\psi\|_{L^{p_3}(Q_i)} \end{aligned}$$

where a generalized Hölder's inequality was applied with

$$p_1 = r$$

and certain $p_2, p_3 \in [r', \infty]$ satisfying

$$\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r'}.$$

We are going to choose such $p_2, p_3 \in [r', \infty]$ and $\alpha \in (0, \beta+1)$ that

$$H^{4\alpha}_p(Q_i) \hookrightarrow L^{p_2}(Q_i), \quad H^{-4\beta}_{p'}(Q_i) \hookrightarrow L^{p_3}(Q_i). \tag{2.6}$$

Note that, if this is the case, then

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} C\phi\psi\Big| &\leq c\|C\|_{L^r_U(\mathbb{R}^N)} \sum_{i\in\mathbb{Z}^N} \|\phi\|_{H^{4\alpha}_p(Q_i)}\|\psi\|_{H^{-4\beta}_{p'}(Q_i)} \\ &\leq c\|C\|_{L^r_U(\mathbb{R}^N)}\Big(\sum_{i\in\mathbb{Z}^N} \|\phi\|^p_{H^{4\alpha}_p(Q_i)}\Big)^{\frac{1}{p}}\Big(\sum_{i\in\mathbb{Z}^N} \|\psi\|^{p'}_{H^{-4\beta}_{p'}(Q_i)}\Big)^{\frac{1}{p'}}. \end{aligned}$$

Consequently, using the estimate

$$\Big(\sum_{i\in\mathbb{Z}^N} \|\phi\|^q_{H^{4\gamma}_q(Q_i)}\Big)^{\frac{1}{q}} \leq c'\|\phi\|_{H^{4\gamma}_q(\mathbb{R}^N)}, \ \gamma \in (0,1), \ q \in (1,\infty),$$

which can be proved analogously as in [4, Lemma 2.4], we obtain

$$\Big|\int_{\mathbb{R}^N} C\phi\psi\Big| \leq \tilde{c}\|C\|_{L^r_U(\mathbb{R}^N)}\|\phi\|_{H^{4\alpha}_p(\mathbb{R}^N)}\|\psi\|_{H^{-4\beta}_{p'}(\mathbb{R}^N)}$$

which gives the result.

From what was said above it is clear that to complete the proof it suffices to show that one can choose the parameters p_2, p_3, α as required in (2.6). Simultaneously we would like to ensure that the set of admissible β's, for which all this can be done, coincides with the interval $I(p)$.

In what follows we are looking for

$$p_2, p_3 \in [r', \infty], \quad \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r'}, \quad 0 < \alpha < \beta + 1, \quad -1 \leq \beta \leq 0 \tag{2.7}$$

which satisfy

$$\alpha-\frac{N}{4p}\geq-\frac{N}{4p_2},\quad -\beta-\frac{N}{4p'}\geq-\frac{N}{4p_3}. \tag{2.8}$$

Note that after adding both inequalities in (2.8) we have $\alpha-\beta-\frac{N}{4}\geq-\frac{N}{4r'}$ so that we will actually consider

$$\alpha\in[\beta+\frac{N}{4r},\beta+1). \tag{2.9}$$

From (2.7), (2.9) we infer that $\frac{N}{4p_3}=\frac{N}{4r'}-\frac{N}{4p_2}$ and $\alpha=\theta(\frac{N}{4r}-1)+\beta+1$ for some $\theta\in(0,1]$, which allows us to write (2.8) as

$$\theta(\frac{N}{4r}-1)+\beta+1-\frac{N}{4p}\geq-\frac{N}{4p_2},\quad -\beta-\frac{N}{4p'}\geq\frac{N}{4p_2}-\frac{N}{4r'} \tag{2.10}$$

or, equivalently,

$$-\frac{N}{4p_2}+\frac{N}{4p}-\frac{N}{4r}\geq\beta\geq-\frac{N}{4p_2}+\theta(1-\frac{N}{4r})-1+\frac{N}{4p}. \tag{2.11}$$

Now, varying p_2 in the interval $[r',\infty]$ and $\theta\in(0,1]$ we observe that on the left hand side of (2.10) we can achieve no more than $\frac{N}{4p}-\frac{N}{4r}$ and that the latter number corresponds to $p_2=\infty$. As for the *infimum* of the right hand side, it will be achieved for $p_2=r'$, $\theta=0$ and thus equal to $-\frac{N}{4r'}-1+\frac{N}{4p}=-1-\frac{N}{4p'}+\frac{N}{4r}=-\beta^*(p')$.

Summarizing we have that, whenever $\beta\in I(p)$, (2.10) can be satisfied for some $p_2\in[r',\infty]$ and $\theta\in(0,1]$. Consequently, whenever $\beta\in I(p)$, (2.8) can be satisfied with some $p_2,p_3\in[r',\infty]$ satisfying $\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r'}$ and with arbitrary $\alpha<\beta+1$ which is close enough to $\beta+1$. On the other hand (2.7)-(2.8) will never hold together if $\beta\notin I(p)$ as, taking into account that $\theta\in(0,1]$, any such β will lie outside the range of the left/right hand sides of (2.11). □

Note that the interval $I(p)$ in the Lemma above describes the set of the admissible β's such that Q_C is a relatively bounded perturbation of the bi-Laplacian in $(H^{-4\beta}_{p'}(\mathbb{R}^N))'$. Thus, note that $0\in I(p)$ only if $\beta^*(p)-1=0$, or in other words, $r\geq p$, which is not assumed to hold in general.

We now consider a perturbation of bi-Laplacian operator with a potential $C\in L^r_U(\mathbb{R}^N)$. Namely, we consider

$$A_C=\Delta^2-C(x)I\ \text{ in } L^p(\mathbb{R}^N)$$

with the domain $D_{L^p}(A_C)$ which will be specified below. Our further concern will be to show that A_C is a negative generator of a C^0 analytic semigroup $\{e^{-A_Ct}:t\geq0\}$ in $L^p(\mathbb{R}^N)$.

Proposition 2.4. *Suppose that $C\in L^r_U(\mathbb{R}^N)$ where $r>\max\{\frac{N}{4},1\}$.*

Then, for each $p\in(1,\infty)$ and any β chosen from the interval

$$\hat{I}(p):=(-\beta^*(p')+1,\beta^*(p)]\subset(0,1], \tag{2.12}$$

A_C with domain $H^{4\beta}_p(\mathbb{R}^N)$ generates a C^0 analytic semigroup in $Y_{p,\beta}:=H^{4(\beta-1)}_p(\mathbb{R}^N)$.

Furthermore, the associated fractional power spaces are given by

$$Y^\zeta_{p,\beta}=\begin{cases}H^{4(\beta+\zeta-1)}_p(\mathbb{R}^N),\ \zeta\in[1-\beta,1],\\ (H^{4(1-\beta-\zeta)}_{p'}(\mathbb{R}^N))',\ \zeta\in[0,1-\beta).\end{cases} \tag{2.13}$$

Proof: We remark that $A+I$ being sectorial in $L^p(\mathbb{R}^N)$ can be also viewed as a sectorial operator in the extrapolated space $\hat{X}^0_p$ with domain $\hat{X}^1_p$. Also, $A+I$ can be viewed as a sectorial operator in $\hat{X}^\beta_p$ with the domain $\hat{X}^{1+\beta}_p$, see V.1.2.6, page 260 in [1].

Now, given $p \in (1,\infty)$, $\beta \in \hat{I}(p)$ and $\alpha < \beta$ which is close enough to β we infer from Lemma 2.3 and (2.5) that

$$\|C\phi\|_{\hat{X}^\beta_p} \le c\|\phi\|_{\hat{X}^{1+\alpha}_p},\ \phi \in \hat{X}^{1+\alpha}_p.$$

Using the interpolation inequality for fractional power spaces (see [9, p. 27]) and Young's inequality we then have

$$\begin{aligned} \|C\phi\|_{\hat{X}^\beta_p} &\le c\|\phi\|_{\hat{X}^{1+\alpha}_p} \le c'\|\phi\|^{\beta-\alpha}_{\hat{X}^\beta_p}\|\phi\|^{1-(\beta-\alpha)}_{\hat{X}^{1+\beta}_p} \\ &\le \epsilon\|\phi\|_{\hat{X}^{1+\beta}_p} + C_\epsilon\|\phi\|_{\hat{X}^\beta_p} \\ &= \epsilon\|(A+I)\phi\|_{\hat{X}^\beta_p} + C_\epsilon\|\phi\|_{\hat{X}^\beta_p},\ \phi \in \hat{X}^{1+\beta}_p,\ \epsilon > 0. \end{aligned} \tag{2.14}$$

Using (2.14) and the above properties of $A+I$ we infer that A_C considered in $\hat{X}^\beta_p$ with the domain $\hat{X}^{1+\beta}_p$ is sectorial in $\hat{X}^\beta_p$ as in this setting it is a relatively bounded perturbation of $A+I$.

On the other hand, coming back to the first line in (2.14) we have the estimate

$$\|C\phi\|_{\hat{X}^\beta_p} \le c'\|\phi\|^{\beta-\alpha}_{\hat{X}^\beta_p}\|\phi\|^{1-(\beta-\alpha)}_{\hat{X}^{1+\beta}_p} = c'\|\phi\|^{\beta-\alpha}_{\hat{X}^\beta_p}\|(A+I)\phi\|^{1-(\beta-\alpha)}_{\hat{X}^\beta_p},\ \phi \in \hat{X}^{1+\beta}_p,$$

which ensures that, for any $\sigma \in (\beta-\alpha,1)$, $C(I+A)^{-\sigma}$ is a bounded operator in $\hat{X}^\beta_p$ (see [9, p. 28-29]). Consequently, the fractional power spaces Y^ζ_p generated by $(A_C, \hat{X}^\beta_p)$ coincide for $\zeta \in [0,1]$ with those generated by $(A+I, \hat{X}^\beta_p)$ (see [9, Theorem 1.4.8]), which proves (2.13). □

Observe that with Proposition 2.4 we obtain below the satement on in part i) of Theorem 1.1. More precisely, we have

Corollary 2.5. *Suppose that $C \in L^r_U(\mathbb{R}^N)$ where $r > \max\{\frac{N}{4},1\}$.*

Then, for each $p \in (1,\infty)$, A_C is a densely defined sectorial operator in $L^p(\mathbb{R}^N)$, that is, $-A_C$ generates in $L^p(\mathbb{R}^N)$ a C^0 analytic semigroup.

Furthermore, if Y^ξ_p, $\xi \ge 0$, denote the fractional power spaces associated to A_C in $L^p(\mathbb{R}^N)$ then

$$Y^\xi_p = H^{4\xi}_p(\mathbb{R}^N),\ \xi \in [0,\beta^*(p)]. \tag{2.15}$$

Proof: We first remark that due to Proposition 2.4 we have that $Y^{1-\beta}_{p,\beta} = L^p(\mathbb{R}^N)$ whenever $\beta \in \hat{I}(p)$ as in (2.12). On the other hand, A_C can be viewed as a sectorial operator in $Y^{1-\beta}_{p,\beta}$ (see (2.13)) with the domain

$$D_{L^p}(A_C) = Y^{2-\beta}_{p,\beta}, \tag{2.16}$$

where β is an arbitrary number from the interval $\hat{I}(p)$. Thus using (2.13) with $\zeta = 1-\beta$ we have that A_C is sectorial in $L^p(\mathbb{R}^N)$. Note that, letting

$$C_\lambda(x) := C(x) - \lambda,\ x \in \mathbb{R}^N, \tag{2.17}$$

with any $\lambda > 0$ sufficiently large, Y_p^ζ are actually the domains of $A_{C_\lambda}^\zeta$ above $L^p(\mathbb{R}^N)$. In particular, $Y_p = Y_p^0 = L^p(\mathbb{R}^N)$ and $D_{L^p}(A_C) = Y_{p,\beta}^{2-\beta}$ coincides as a set with $A_{C_\lambda}^{-1}(L^p(\mathbb{R}^N)) = A_{C_\lambda}^{-1}(Y_{p,\beta}^{1-\beta})$.

Since $Y_p = L^p(\mathbb{R}^N) = Y_{p,\beta}^{1-\beta}$, then using (2.13) with $\zeta = 1-\beta+\xi$ we obtain (2.15). □

Under the additional assumption that $p \le r$ Corollary 2.5 applies with $\beta = 1$ as in this case $\beta^*(p) = 1$. In particular, we have the following result.

Corollary 2.6. *Suppose that $C \in L_U^r(\mathbb{R}^N)$ with $r > \max\{\frac{N}{4}, 1\}$.*

Then, for any $1 < p \le r$,

i) $D_{L^p}(A_C) = H_p^4(\mathbb{R}^N)$ and A_C with this domain is sectorial in $L^p(\mathbb{R}^N)$,

ii) the fractional power spaces Y_p^ξ generated by $(A_C, L^p(\mathbb{R}^N))$ coincide with $H_p^{4\xi}(\mathbb{R}^N)$ for each $\xi \in [0,1]$.

Now, following again [1, p. 262] we can construct the extrapolated fractional power spaces for A_C. For this, choosing in (2.17) any sufficiently large $\lambda > 0$, we denote by $(Y_p)^{-1}$ the completion of $Y_p = L^p(\mathbb{R}^N)$ under the $\|A_{C_\lambda}^{-1} \cdot \|_{Y_p}$-norm, which is so called extrapolated space of Y_p generated by A_C. Then A_C extends uniquely to sectorial operator $A_C : Y_p \to (Y_p)^{-1}$. We then consider the one-sided fractional power scale

$$\hat{Y}_p^\alpha := \big((Y_p)^{-1}\big)^\alpha, \ \alpha \ge 0,$$

and define the extrapolated fractional power scale of order 1 generated by (A_C, Y_p) letting

$$E_p^\alpha := \hat{Y}_p^{\alpha+1}, \ \alpha \ge -1.$$

Note that $E_p^{-\alpha}$ can be viewed for $\alpha \in (0,1]$ as a completion of $Y_p = L^p(\mathbb{R}^N)$ with respect to the $\|A_{C_\lambda}^{-\alpha} \cdot \|_{Y_p}$-norm. Also note that in the particular case $C = 0$ the above procedure has already been carried out in (2.4).

We now obtain the characterization of the spaces E_p^α as stated in Theorem 1.1. In particular we show that the E_p^α spaces coincide with the fractional power spaces associated to $A = \Delta^2$, for some range of α.

Corollary 2.7. *Suppose that $C \in L_U^r(\mathbb{R}^N)$ with $r > \max\{\frac{N}{4}, 1\}$.*

Then, given any $p \in (1,\infty)$, for the two sided fractional power scale generated by (Y_p, A_C) we have that

$$E_p^\alpha = \begin{cases} H_p^{4\alpha}(\mathbb{R}^N) & \text{for } \ \beta^*(p) \ge \alpha \ge 0, \\ (H_{p'}^{-4\alpha}(\mathbb{R}^N))' & \text{for } \ -\beta_*(p) \le \alpha < 0, \end{cases} \tag{2.18}$$

where $\beta_*(p) = \beta^*(p')$.

On this scale of spaces, the analytic semigroup generated by $-A_C$ satisfies, for any $-\beta_*(p) \le \sigma \le \xi \le \beta^*(p)$

$$\|e^{-A_C t}\|_{\mathcal{L}(E_p^\sigma, E_p^\xi)} \le M \frac{e^{-\omega t}}{t^{\xi-\sigma}}, \quad t > 0,$$

for some $M \ge 1$ and $\omega \in \mathbb{R}$.

All the above results hold with $\beta^*(p) = 1$ whenever $1 < p \le r$, with $\beta_*(p) = 1$ whenever $p' \le r$ and with $\beta^*(p) = \beta_*(p) = 1$ if $p, p' \le r$. In particular, the results hold with $\beta^*(p) = \beta_*(p) = 1$ for any $p \in (1,\infty)$ if $C \in L^\infty(\mathbb{R}^N)$.

Proof: The proof follows from Corollaries 2.5, 2.6 and from the general results in [1, Theorem V.1.4.12] and [9, Theorem 1.4.3]. $\square$

Remark 2.8. *Note that in the above results it is implicitly included that $-A_C$ generates a C^0 analytic semigroup on $(H_{p'}^{4\beta^*(p')}(\mathbb{R}^N))'$. Certainly, this could hardly be concluded in Proposition 2.4 due to the 'relative boundedness' technique used therein, which did not work well for the left extreme of the interval $\hat{I}(p)$. Nonetheless, $(H^{4\beta^*(p')}(\mathbb{R}^N))'$ can be viewed as a fractional power spaces $\hat{Y}_p^{\alpha+1}$ whereas A_C is sectorial in $\hat{Y}_p^{\sigma}$ for every $\sigma \geq 0$.*

Since we now turn our attention to the Hilbert space case, setting $p = 2$, we first recall that

$$\beta^*(2) > \frac{1}{2}. \tag{2.19}$$

From Corollary 2.6 if $C \in L_U^r(\mathbb{R}^N)$ with $r > \frac{N}{4}$ and $r \geq 2$, then $D_{L^2}(A_C) = H^4(\mathbb{R}^N)$ and A_C with this domain is a symmetric operator in $L^2(\mathbb{R}^N)$. In what follows we prove that A_C in this case is also bounded below and that actually all these hold also when $\max\{\frac{N}{4}, 1\} < r < 2$, in which case $D_{L^2}(A_C)$ is characterized as in (2.16).

Lemma 2.9. *Suppose that $C \in L_U^r(\mathbb{R}^N)$ with $r > \max\{\frac{N}{4}, 1\}$.*

Then the domain of the operator A_C in $L^2(\mathbb{R}^N)$, $D_{L^2}(A_C)$, is contained in $H^2(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} A_C\phi\psi = \int_{\mathbb{R}^N} \Delta\phi\Delta\psi - \int_{\mathbb{R}^N} C(x)\phi\psi = \int_{\mathbb{R}^N} \phi A_C\psi, \ \ \phi, \psi \in D_{L^2}(A_C). \tag{2.20}$$

Furthermore, for any $\varepsilon \in (0, 1)$ there exists a certain $c_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^N} (|\Delta\phi|^2 - C(x)\phi^2) \geq (1 - \varepsilon)\|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} - c_\varepsilon\|\phi\|^2_{L^2(\mathbb{R}^N)} \ \ \textit{for each} \ \ \phi \in H^2(\mathbb{R}^N).$$

In particular, there exists $\omega_0 \in \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} (|\Delta\phi|^2 - C(x)\phi^2) \geq \omega_0\|\phi\|^2_{L^2(\mathbb{R}^N)} \ \ \textit{for each} \ \ \phi \in H^2(\mathbb{R}^N). \tag{2.21}$$

Proof: Note that using Corollary 2.5 and $\beta^*(2) > \frac{1}{2}$ we immediately have that $D_{L^2}(A_C) \subset Y_2^{\frac{1}{2}} = H^2(\mathbb{R}^N)$.

In the proof of (2.20) it actually suffices to show the first equality and focus on the case when $\max\{\frac{N}{4}, 1\} < r < 2$ as otherwise we know that $D_{L^2}(A_C) = H^4(\mathbb{R}^N)$ and the result follows easily.

Hence, we have $H_r^4(\mathbb{R}^N) \subset H^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \subset L^{r'}(\mathbb{R}^N)$. On the other hand, Proposition 2.1 and Corollary 2.6 with $p = r$ imply that A and Q_C are continuous from $H_r^4(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$. Therfore for $\phi, \psi \in H_r^4(\mathbb{R}^N)$, $\int_{\mathbb{R}^N} \Delta\phi\Delta\psi$ and $\int_{\mathbb{R}^N} C(x)\phi\psi$ are well defined.

Consequently, taking first $\phi, \psi \in C_0^\infty(\mathbb{R}^N)$ and then using a density argument we obtain

$$\int_{\mathbb{R}^N} A_C\phi\psi = \int_{\mathbb{R}^N} \Delta\phi\Delta\psi - \int_{\mathbb{R}^N} C(x)\phi\psi, \quad \phi, \psi \in H_r^4(\mathbb{R}^N). \tag{2.22}$$

In what follows we will extend this to $\phi, \psi \in D_{L^2}(A_C)$.

Observe that $A_{C_\lambda}^{-1}(C_0^\infty(\mathbb{R}^N))$ is dense in $D_{L^2}(A_C)$, with the graph norm, since for λ sufficiently large, $D_{L^2}(A_C) = A_{C_\lambda}^{-1}(L^2(\mathbb{R}^N))$ and $C_0^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$. Also, from Corollary 2.6 $A_{C_\lambda}^{-1}(C_0^\infty(\mathbb{R}^N))$ is contained in $H_r^4(\mathbb{R}^N)$ and then (2.22) actually holds for ϕ, ψ from a dense subset of $D_{L^2}(A_C)$ and thus for any $\phi, \psi \in D_{L^2}(A_C)$.

It remains to prove (2.21) for which we will consider open disjoint cubes $Q_i \subset \mathbb{R}^N$ centered at $i \in \mathbb{Z}^N$ with all its edges unitary and parallel to the axes. We assume below that $r < \infty$ and the proof goes with minor changes if $r = \infty$.

Letting $r' = \frac{r}{r-1}$ and $\mathbb{R}^N = \cup_{i\in\mathbb{Z}^N}\overline{Q_i}$ we then have

$$\begin{aligned}\left|\int_{\mathbb{R}^N} C(x)|\phi|^2\right| &\le \sum_{i\in\mathbb{Z}^N}\int_{Q_i}|C(x)||\phi|^2 \le \sum_{i\in\mathbb{Z}^N}\|C\|_{L^r(Q_i)}\|\phi\|^2_{L^{2r'}(Q_i)} \\ &\le \|C\|_{L_U^r(\mathbb{R}^N)}\sum_{i\in\mathbb{Z}^N}\|\phi\|^2_{H^{2s}(Q_i)},\end{aligned} \tag{2.23}$$

where $s \in (0,1)$ is chosen such that $H^{2s}(Q_i) \hookrightarrow L^{2r'}(Q_i)$, i.e. $2s - \frac{N}{2} \ge -\frac{N}{2r'}$. By interpolation

$$\|\phi\|_{H^{2s}(Q_i)} \le c\|\phi\|^s_{H^2(Q_i)}\|\phi\|^{1-s}_{L^2(Q_i)}, \ \phi \in H^2(Q_i),$$

(see [14, §2.4.2(11)]) and hence for each $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} C(x)|\phi|^2\right| &\le \sum_{i\in\mathbb{Z}^N}(\varepsilon\|\phi\|^2_{H^2(Q_i)} + c_\varepsilon\|\phi\|^2_{L^2(Q_i)}) \\ &= \varepsilon\|\phi\|^2_{H^2(\mathbb{R}^N)} + c_\varepsilon\|\phi\|^2_{L^2(\mathbb{R}^N)}, \ \phi \in H^2(\mathbb{R}^N).\end{aligned} \tag{2.24}$$

Since the norms $\|(-\Delta + I)\phi\|_{L^2(\mathbb{R}^N)}$ and $\|\phi\|_{H^2(\mathbb{R}^N)}$ are equivalent (see [14, §2.5.3 Step 3]) we get the result. □

Note that (2.24) implies that $C\phi^2 \in L^1(\mathbb{R}^N)$ if $\phi \in H^2(\mathbb{R}^N)$ as stated in Remark 1.2 iv) of the introduction.

We now conclude the last statement of Theorem 1.1.

Corollary 2.10. *Suppose that $C \in L_U^r(\mathbb{R}^N)$ with $r > \max\{\frac{N}{4}, 1\}$. Then*
(i) A_C is a selfadjoint operator in $L^2(\mathbb{R}^N)$ and
(ii) the analytic semigroup generated by $-A_C$ in $L^2(\mathbb{R}^N)$ is exponentially asymptotically decaying if and only if (2.21) holds with a certain $\omega_0 > 0$.

Proof: We know that, for sufficiently large $\lambda > 0$, A_{C_λ} is surjective from $D_{L^2}(A_C)$ onto $L^2(\mathbb{R}^N)$. On the other hand Lemma 2.9 ensures that A_{C_λ} is a symmetric operator in $L^2(\mathbb{R}^N)$. Consequently, A_{C_λ} and hence also A_C is a selfadjoint operator in $L^2(\mathbb{R}^N)$.

Now, if (2.21) holds with $\omega_0 > 0$ then the semigroup $\{e^{-A_C t} : t \ge 0\}$ decays exponentially since the spectrum $\sigma(A_C)$ is contained in the interval $[\omega_0, \infty)$.

Conversely, if $\{e^{-A_C t} : t \ge 0\}$ decays exponentially, that is if $\|e^{-A_C t}\|_{\mathcal{L}(L^2(\mathbb{R}^N))} \le Me^{-\omega t}$ for some $\omega > 0$, then Theorem 5.3 in [12] and the selfadjointness of A_C imply that $\sigma(A_C) > 0$. Using the characterization of the fractional power space Y_p^ξ for $p = 2$, $\xi = \frac{1}{2}$ (see Corollary 2.5 and (2.19)) we infer that $Y_2^{\frac{1}{2}}$ coincides with $H^2(\mathbb{R}^N)$; in particular, the norms $\|A_C^{\frac{1}{2}}\phi\|_{L^2(\mathbb{R}^N)}$

and $\|\phi\|_{H^2(\mathbb{R}^N)}$ are equivalent. Also, since fractional powers of a positive selfadjoint operator are selfadjoint (see [9, p. 27]), we infer that

$$\|A_C^{\frac{1}{2}}\phi\|^2_{L^2(\mathbb{R}^N)} = \langle A_C\phi, \phi\rangle_{L^2(\mathbb{R}^N)},\ \phi \in D_{L^2}(A_C). \tag{2.25}$$

From this and (2.20) we have

$$\begin{aligned}\|\phi\|^2_{L^2(\mathbb{R}^N)} \leq c\|A_C^{\frac{1}{2}}\phi\|^2_{L^2(\mathbb{R}^N)} &= c\langle A_C\phi, \phi\rangle_{L^2(\mathbb{R}^N)} \\ &= c\int_{\mathbb{R}^N}(|\Delta\phi|^2 - C(x)\phi^2),\ \phi \in D_{L^2}(A_C).\end{aligned} \tag{2.26}$$

We also remark that in a similar way to (2.23)-(2.24) one can obtain

$$\left|\int_{\mathbb{R}^N} C(x)\phi\psi\right| \leq M\|\phi\|_{H^2(\mathbb{R}^N)}\|\psi\|_{H^2(\mathbb{R}^N)},\ \phi,\psi \in H^2(\mathbb{R}^N).$$

Hence, using (2.26) and density of $D_{L^2}(A_C)$ in $Y_2^{\frac{1}{2}} = H^2(\mathbb{R}^N)$, we infer that (2.21) holds with $\omega_0 > 0$. □

Finally we show some continuity of the constant ω_0 in Lemma 2.9. Note that in fact the constant ω_0 in (1.10), or in (2.21), gives a lower bound of the bottom spectrum of A_C in $L^2(\mathbb{R}^N)$. So the result below is a sort of continuity of the bottom spectrum with respect to the diffusion coefficient.

Corollary 2.11. *Suppose that $C \in L^r_U(\mathbb{R}^N)$ with $r > \max\{\frac{N}{4}, 1\}$ and let $\omega_0 \in \mathbb{R}$ be as in (2.21).*

Then there is a continuous decreasing real valued function $\omega(\nu)$ defined in a certain interval $[0,\nu_0]$ such that

$$\int_{\mathbb{R}^N}\big((1-\nu)|\Delta\phi|^2 - C(x)\phi^2\big) \geq \omega(\nu)\int_{\mathbb{R}^N}\phi^2 \ \ \textit{for all} \ \ \phi \in H^2(\mathbb{R}^N),\ \nu \in [0,\nu_0],$$

and

$$\lim_{\nu\to 0^+}\omega(\nu) = \omega(0) = \omega_0.$$

Proof: We write

$$\int_{\mathbb{R}^N}\big((1-\nu)|\Delta\phi|^2 - C(x)\phi^2\big) = (1-\nu)\int_{\mathbb{R}^N}|\Delta\phi|^2 - (1-2\nu)\int_{\mathbb{R}^N}C(x)|\phi|^2 - 2\nu\int_{\mathbb{R}^N}C(x)|\phi|^2$$

and then using (2.24) with $\varepsilon = 1/2$ in the last term we get

$$\int_{\mathbb{R}^N}\big((1-\nu)|\Delta\phi|^2 - C(x)\phi^2\big) \geq (1-2\nu)\int_{\mathbb{R}^N}\big(|\Delta\phi|^2 - C(x)\phi^2\big) - \nu c\|\phi\|^2_{L^2(\mathbb{R}^N)}.$$

Hence, using (2.21) we get

$$\int_{\mathbb{R}^N}\big((1-\nu)|\Delta\phi|^2 - C(x)\phi^2\big) \geq \big(\omega_0(1-2\nu) - \nu c\big)\|\phi\|^2_{L^2(\mathbb{R}^N)}.$$

Letting $\omega(\nu) = \big(\omega_0(1-2\nu) - \nu c\big)$ we obtain the result. □

3. Local well posedness for nonlinear problems

In this section we prove that there is a unique solution of (1.1) continuously depending on the initial condition, where $u_0 \in L^p(\mathbb{R}^N)$ or $u_0 \in H^2_p(\mathbb{R}^N)$, respectively and $1 < p < \infty$. In particular we will prove Theorems 1.3 and 1.4.

We will consider (1.1) rewritten as

$$\begin{cases} \dot{u} + A_m u = f_0(\cdot, u) + g =: \mathcal{F}(u), \ t > 0, \\ u(0) = u_0 \in L^p(\mathbb{R}^N) \text{ or } H^2_p(\mathbb{R}^N), \end{cases} \tag{3.1}$$

where

$$A_m := A - m(x)I,$$

for which the results of Section 2 apply because of (1.11). In particular we will use below the scale of spaces, E^α_p, $-\beta^*(p') \leq \alpha \leq \beta^*(p)$, as in (2.18) and the smoothing properties of the semigroup generated by $-A_m$ on this scale, as in Corollary 2.7.

We start from the following technical lemma that will be used below.

Lemma 3.1. *If f_0 satisfies (1.3), (1.5) and $f_0(\cdot, 0) = 0$ then there exists a decomposition*

$$f_0(x, v) = f_{01}(x, v) + f_{02}(x, v), \ x \in \mathbb{R}^N, \ v \in \mathbb{R},$$

where

$$f_{01}(x, 0) = f_{02}(x, 0) = 0,$$

$$f_{01} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \ \ \textit{is a globally Lipschitz map}$$

and

$$|f_{02}(x, v_1) - f_{02}(x, v_2)| \leq c|v_1 - v_2|(|v_1|^{\rho-1} + |v_2|^{\rho-1}), \ v_1, v_2 \in \mathbb{R}, \tag{3.2}$$

for some $c > 0$.

Proof: Define

$$f_{01}(x, v) = \begin{cases} f_0(x, v), \ x \in \mathbb{R}, \ |v| \leq 1, \\ f_0(x, 1), \ x \in \mathbb{R}, \ |v| > 1, \end{cases}$$

and

$$f_{02}(x, v) = f_0(x, v) - f_{01}(x, v), \ x \in \mathbb{R}^N, \ v \in \mathbb{R}.$$

With the aid of (1.3), choosing $L_0 > 0$ as a Lipschitz constant for f_0 restricted to $\mathbb{R}^N \times [-1, 1]$, we have that

$$|f_{01}(x, v_1) - f_{01}(x, v_2)| \leq L_0|v_1 - v_2|, \ x \in \mathbb{R}^N, \ v_1, v_2 \in \mathbb{R}.$$

Using the above relations and (1.5) we obtain (3.2). □

Below we will prove Theorems 1.3 and 1.4 for which we will use the analytic semigroup approach as in [9]. Since we deal with critical exponents we will actually use the extension of this approach developed in [3].

3.1. **Local well posedness of (1.1) in $L^p(\mathbb{R}^N)$.** In this subsection we prove the existence of solutions of (1.1) with initial data $u_0 \in L^p(\mathbb{R}^N)$, for which we will assume that the condition (1.5) holds with some $1 < \rho \le 1 + \frac{4p}{N} =: \rho_c^1$. We remark that the solutions will satisfy the variation of constants formula (1.14) and will possess appropriate regularity properties; namely (1.15) holds.

Furthermore, whenever $\rho < 1 + \frac{4p}{N} = \rho_c^1$, the solutions satisfy (1.17) In particular, an $L^p(\mathbb{R}^N)$-estimate on compact time intervals will guarantee that the solution exists globally for $t \ge 0$ (see [3]; also [6, Corollary 1.1]).

Proof of Theorem 1.3. Recalling the formulation of the problem as in (3.1) and following the approach in [3] all what needs to be shown is that $\mathcal{F}$ in (3.1) is an ε-regular map relative to the pair of spaces from the fractional power scale associated with the main part operator. This, in terms of E_p^α-scale and for the case when initial data are in $L^p(\mathbb{R}^N)$, translates into the requirement that the condition

$$\|\mathcal{F}(v) - \mathcal{F}(w)\|_{E_p^{\gamma(\varepsilon)-1}} \le c\|v-w\|_{E_p^\varepsilon}\big(1 + \|v\|_{E_p^\varepsilon}^{\rho-1} + \|w\|_{E_p^\varepsilon}^{\rho-1}\big),\ v, w \in E_p^\varepsilon, \tag{3.3}$$

holds for certain constants $c > 0$, $\varepsilon \in (0, \frac{1}{\rho})$, and $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$.

Actually, due to Lemma 3.1, it is sufficient to show that there are constants $c > 0$ and

$$0 < \varepsilon < 1, \quad \rho\varepsilon \le \gamma(\varepsilon) < 1 \tag{3.4}$$

such that

$$\|f_{02}(v) - f_{02}(w)\|_{E_p^{\gamma(\varepsilon)-1}} \le c\|v-w\|_{E_p^\varepsilon}\big(\|v\|_{E_p^\varepsilon}^{\rho-1} + \|w\|_{E_p^\varepsilon}^{\rho-1}\big),\ v, w \in E_p^\varepsilon. \tag{3.5}$$

Observe that Corollary 2.7 yields

$$\begin{aligned} &E_p^\varepsilon = H_p^{4\varepsilon}(\mathbb{R}^N),\ \varepsilon \in [0, \beta^*(p)],\\ &E_p^{\gamma(\varepsilon)-1} = (H_{p'}^{4(1-\gamma(\varepsilon))}(\mathbb{R}^N))',\ \gamma(\varepsilon) \in [1-\beta^*(p'), 1) \end{aligned} \tag{3.6}$$

and hence we have

$$\begin{aligned} &E_p^\varepsilon \hookrightarrow L^s(\mathbb{R}^N),\ \alpha \in [0, \beta^*(p)],\ 4\varepsilon - \frac{N}{p} \ge -\frac{N}{s},\ s \ge p,\\ &E_p^{\gamma(\varepsilon)-1} \hookleftarrow L^\sigma(\mathbb{R}^N),\ \gamma(\varepsilon) \in [1-\beta^*(p'), 1),\ \frac{Np}{N + 4(1-\gamma(\varepsilon))p} \le \sigma \le p,\ \sigma > 1. \end{aligned} \tag{3.7}$$

Note that the second embedding in (3.7) holds with $\sigma = \frac{Np}{N+4(1-\gamma(\varepsilon))p} > 1$ provided that

$$1 > \gamma(\varepsilon) \ge 1 - \beta^*(p') =: \hat{\gamma} \ \text{ and } \ \gamma(\varepsilon) > \frac{N + 4p - Np}{4p} =: \tilde{\gamma}. \tag{3.8}$$

Using this we have

$$\begin{aligned} &\|f_{02}(v) - f_{02}(w)\|_{E_p^{\gamma(\varepsilon)-1}} \le c'\|f_{02}(v) - f_{02}(w)\|_{L^{\frac{Np}{N+4(1-\gamma(\varepsilon))p}}(\mathbb{R}^N)}\\ &\le c'\|\hat{c}|v-w|(|v|^{\rho-1} + |w|^{\rho-1})\|_{L^{\frac{Np}{N+4(1-\gamma(\varepsilon))p}}(\mathbb{R}^N)}\\ &\le c''\left(\int_{\mathbb{R}^N} |v-w|^{\frac{Np}{N+4(1-\gamma(\varepsilon))p}}\big(|v|^{\frac{Np(\rho-1)}{N+4(1-\gamma(\varepsilon))p}} + |w|^{\frac{Np(\rho-1)}{N+4(1-\gamma(\varepsilon))p}}\big)dx\right)^{\frac{N+4(1-\gamma(\varepsilon))p}{Np}}. \end{aligned}$$

Applying next Hölder's inequality with

$$\hat{\theta} = \frac{N + 4(1-\gamma(\varepsilon))p}{N - 4p\varepsilon}, \quad \hat{\theta}' = \frac{\hat{\theta}}{\hat{\theta} - 1} = \frac{N + 4(1-\gamma(\varepsilon))p}{4(1-\gamma(\varepsilon))p + 4p\varepsilon},$$

and thus assuming that

$$\frac{N}{4p} > \varepsilon, \tag{3.9}$$

we obtain

$$\begin{aligned} &\|f_{02}(v) - f_{02}(w)\|_{E_p^{\gamma(\varepsilon)-1}} \\ &\qquad \le c'''\|v - w\|_{L^{\frac{Np}{N-4p\varepsilon}}(\mathbb{R}^N)} \Big(\|v\|^{\rho-1}_{L^{\frac{N(\rho-1)}{4(1-\gamma(\varepsilon))+4\varepsilon}}(\mathbb{R}^N)} + \|w\|^{\rho-1}_{L^{\frac{N(\rho-1)}{4(1-\gamma(\varepsilon))+4\varepsilon}}(\mathbb{R}^N)} \Big). \end{aligned} \tag{3.10}$$

The right hand side of (3.10) can be bounded by the right hand side of (3.5) provided that

$$H_p^{4\varepsilon}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np}{N-4p\varepsilon}}(\mathbb{R}^N) \tag{3.11}$$

and

$$H_p^{4\varepsilon}(\mathbb{R}^N) \hookrightarrow L^{\frac{N(\rho-1)}{4(1-\gamma(\varepsilon))+4\varepsilon}}(\mathbb{R}^N), \tag{3.12}$$

where we also have the limitation

$$\varepsilon \le \beta^*(p). \tag{3.13}$$

Observe that (3.13) is true whenever (3.9) is satisfied. Indeed, if $\beta^*(p) < 1$ we have it because $\beta^*(p) > \frac{N}{4p}$ whereas if $\beta^*(p) = 1$ this is a consequence of the restriction $\varepsilon < 1$.

On the other hand, assuming (3.9) note that (3.11) holds true, whereas for (3.12) one needs

$$\overline{\gamma} := \frac{(4\varepsilon p - N)(\rho - 1) + 4(1+\varepsilon)p}{4p} \ge \gamma(\varepsilon) \ge \frac{4(1+\varepsilon)p - N(\rho-1)}{4p} =: \underline{\gamma}. \tag{3.14}$$

We remark that for $\rho \in (1, 1 + \frac{4p}{N}]$ and $\varepsilon > 0$ we have $\overline{\gamma} > \underline{\gamma} > 0$ and $\overline{\gamma} \ge \varepsilon\rho$. We also have $1 > \overline{\gamma}$ if $\varepsilon \in (0, \frac{N(\rho-1)}{4p\rho})$. Furthermore, $\overline{\gamma} > \tilde{\gamma}$ if and only if $\varepsilon > \max\{0, \frac{N(\rho-p)}{4p\rho}\}$ and $\bar{\gamma} > \hat{\gamma}$ if $\varepsilon > \frac{N(\rho-1)}{4p\rho}$.

The set of $(\rho, \varepsilon, \gamma(\varepsilon))$ solving (3.4), (3.8), (3.9) and (3.14) is thus nonempty and contains triples $(\rho, \varepsilon, \gamma(\varepsilon))$, where $\rho \in (1, 1 + \frac{4p}{N}]$, $\varepsilon \in (\max\{0, \frac{N(\rho-p)}{4p\rho}\}, \frac{N(\rho-1)}{4p\rho})$ and

$$\gamma(\varepsilon) \in [\rho\varepsilon, \overline{\gamma}] \cap [\underline{\gamma}, \overline{\gamma}] \cap (\tilde{\gamma}, \overline{\gamma}] \cap [\hat{\gamma}, \bar{\gamma}] =: \mathcal{I}(\varepsilon).$$

Actually, for admissible $(\rho, \varepsilon, \gamma(\varepsilon))$ the left hand side inequality in (3.14) implies

$$\rho \le \frac{N + 4p - 4p\gamma(\varepsilon)}{N - 4p\varepsilon}$$

and via (3.4) we then have

$$\rho \le \frac{N + 4p - 4p\rho\varepsilon}{N - 4p\varepsilon}.$$

The second of these inequalities holds if and only if $\rho \le \frac{N+4p}{N} = \rho_c^1$. The first one shows that $\rho = \rho_c^1$ cannot be attained for any $\gamma(\varepsilon) > \rho_c^1\varepsilon$. Thus $\rho = \rho_c^1$ necessitates $\gamma(\varepsilon) = \varepsilon\rho_c^1$, in which case we have $\overline{\gamma} = \varepsilon\rho_c^1$; that is, if $\rho = \rho_c^1$, $\mathcal{I}(\varepsilon) = \{\varepsilon\rho_c^1\}$.

Therefore (3.5) and (3.3) are satisfied. We thus have that $\mathcal{F}$ in (3.1) is an ε-regular map relative to the pair of spaces (E_p^0, E_p^{-1}). Thus [3, Corollary 1] ensures that (1.1) is locally well posed in E_p^0 and, in addition,

$$u \in C([0,\tau_0), E_p^0) \cap C((0,\tau_0), E_p^{\gamma(\varepsilon)}) \cap C^1((0,\tau_0), E_p^{\gamma(\varepsilon)-1}). \tag{3.15}$$

Since the above analysis shows that $\gamma(\varepsilon)$ can be chosen arbitrarily less than $\overline{\gamma}(\rho, \frac{N(\rho-1)}{4p\rho}) = 1$ we infer from (3.15) and (3.6) that

$$u \in C((0,\tau_0), H_p^{4\theta}(\mathbb{R}^N)) \cap C^1((0,\tau_0), H_p^{4\theta}(\mathbb{R}^N)), \text{ whenever } \theta \leq \beta^*(p) \text{ and } \theta < 1. \tag{3.16}$$

Hence, if $\beta^*(p) < 1$, (3.16) gives (1.15).

If $\beta^*(p) = 1$ then from (3.16) we infer that $u \in H_p^2(\mathbb{R}^N)$. Using Theorem 1.4 (see (3.29) below), after restarting the solution at any positive time of its existence, we then obtain (1.15). Note that if $2 < \frac{N}{p}$, then Theorem 1.4 iii) applies as $\rho_c^1 < \rho_c^2$. On the other hand, if $2 \geq \frac{N}{p}$ Theorem 1.4 ii) or iii) apply as well because we are dealing now with the case $\beta^*(p) = 1$ so that $r \geq p$. □

Remark 3.2. *Due to the above consideration it suffices to assume that g belongs to $E_p^{\gamma(\varepsilon)-1}$ for arbitrarily fixed $\gamma(\varepsilon) \in \mathcal{I}(\varepsilon)$, $\varepsilon \in (\max\{0, \frac{N(\rho-p)}{4p\rho}\}, \frac{N(\rho-1)}{4p\rho})$, and the problem (1.1) remains well posed in $L^p(\mathbb{R}^N)$ with $\rho \in (1, \rho_c^1]$. Nonetheless the solution will not be as regular as stated in (1.15).*

3.2. **Local well posedness of (1.1) in $H_p^2(\mathbb{R}^N)$.** We proceed to the proof of Theorem 1.4. First note that under the assumptions of the Theorem we have $\beta^*(p) > \frac{1}{2}$ so that from the results in Corollary 2.7, in case (i) (resp. (ii)) of Theorem 1.4, we have the embedding $E_p^{\frac{1}{2}} \hookrightarrow L^\infty(\mathbb{R}^N)$ (resp. $E_p^{\frac{1}{2}} \hookrightarrow L^q(\mathbb{R}^N)$ for $q \in [p,\infty)$). Therefore, the Nemytskiĭ map $\mathcal{F}$ is Lipschitz continuous on bounded sets from $E_p^{\frac{1}{2}}$ into E_p. Consequently, both cases (i) and (ii) in Theorem 1.4, as well as (1.16) follow from [9, Theorem 3.3.3].

On the other hand, in case (iii), we will use the approach of [3, Corollary 1]. The reason for this is that now $\mathcal{F}$ may not take $H_p^2(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$ unless $\rho \leq \frac{N}{N-2p}$. Actually note that if $1 < \rho \leq \frac{N}{N-2p}$ the map $\mathcal{F}$ is Lipschitz continuous on bounded sets from $H_p^2(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$; see Proposition 3.4. So this case follows again from [9, Theorem 3.3.3].

Proof of Theorem 1.4. As mentioned above we only consider case (iii) here focusing on the situation when

$$\rho \in (\frac{N}{N-2p}, \rho_c^2]. \tag{3.17}$$

Following [3] we will show that $\mathcal{F}$ is an ε-regular map relative to the pair of spaces $(E_p^{\frac{1}{2}}, E_p^{-\frac{1}{2}})$; namely, there are constants $c > 0$, $\varepsilon \in (0, \frac{1}{\rho})$, and $\gamma(\varepsilon) \in [\rho\varepsilon, 1)$ such that

$$\|\mathcal{F}(v) - \mathcal{F}(w)\|_{E_p^{\gamma(\varepsilon)-\frac{1}{2}}} \leq c\|v-w\|_{E_p^{\frac{1}{2}+\varepsilon}}\big(1 + \|v\|^{\rho-1}_{E_p^{\frac{1}{2}+\varepsilon}} + \|w\|^{\rho-1}_{E_p^{\frac{1}{2}+\varepsilon}}\big), \ v, w \in E_p^{\frac{1}{2}+\varepsilon}. \tag{3.18}$$

Note that, using Lemma 3.1, it is actually sufficient to show that (3.18) holds for $\mathcal{F} = f_{02}$.

Observe that Corollary 2.7 yields

$$
\begin{aligned}
&E_p^{\frac{1}{2}+\varepsilon} = H_p^{2+4\varepsilon}(\mathbb{R}^N),\ \beta^*(p) \geq \frac{1}{2} + \varepsilon \geq 0, \\
&E_p^{\gamma(\varepsilon)-\frac{1}{2}} = (H_{p'}^{2-4\gamma(\varepsilon)}(\mathbb{R}^N))',\ -\beta^*(p') \leq \gamma(\varepsilon) - \frac{1}{2} < 0
\end{aligned}
\tag{3.19}
$$

and hence we have

$$
\begin{aligned}
&E_p^{\frac{1}{2}+\varepsilon} \hookrightarrow L^s(\mathbb{R}^N),\ \beta^*(p) \geq \frac{1}{2} + \varepsilon \geq 0,\ 2 + 4\varepsilon - \frac{N}{p} \geq -\frac{N}{s},\ s \geq p, \\
&E_p^{\gamma(\varepsilon)-\frac{1}{2}} \hookleftarrow L^\sigma(\mathbb{R}^N),\ \gamma(\varepsilon) \in [\frac{1}{2} - \beta^*(p'), \frac{1}{2}],\ \frac{Np}{N + (2-4\gamma(\varepsilon))p} \leq \sigma \leq p,\ \sigma > 1.
\end{aligned}
\tag{3.20}
$$

Given $\rho \in (1, \frac{N+2p}{N-2p}]$ and some suitable

$$
0 < \varepsilon < 1, \quad \rho\varepsilon \leq \gamma(\varepsilon) < 1 \tag{3.21}
$$

from the second embedding in (3.20) we have

$$
\begin{aligned}
&\|f_{02}(v) - f_{02}(w)\|_{E_p^{\gamma(\varepsilon)-\frac{1}{2}}} \leq c'\|f_{02}(v) - f_{02}(w)\|_{L^{\frac{Np}{N+(2-4\gamma(\varepsilon))p}}(\mathbb{R}^N)} \\
&\leq c'\| |v-w|(|v|^{\rho-1} + |w|^{\rho-1})\|_{L^{\frac{Np}{N+(2-4\gamma(\varepsilon))p}}(\mathbb{R}^N)} \\
&\leq c''\left(\int_{\mathbb{R}^N} |v-w|^{\frac{Np}{N+(2-4\gamma(\varepsilon))p}} \left(|v|^{\frac{Np(\rho-1)}{N+(2-4\gamma(\varepsilon))p}} + |w|^{\frac{Np(\rho-1)}{N+(2-4\gamma(\varepsilon))p}}\right) dx\right)^{\frac{N+(2-4\gamma(\varepsilon))p}{Np}},
\end{aligned}
\tag{3.22}
$$

where $N + (2-4\gamma(\varepsilon))p > 0$ for $\gamma(\varepsilon) < 1$ as $N > 2p$. Using next Hölder's inequality with

$$
\theta = \frac{N + (2-4\gamma(\varepsilon))p}{N - 2p - 4p\varepsilon}, \quad \theta' = \frac{\theta}{\theta - 1} = \frac{N + (2-4\gamma(\varepsilon))p}{4(1+\varepsilon-\gamma(\varepsilon))p},
$$

we obtain

$$
\begin{aligned}
&\|f_{02}(v) - f_{02}(w)\|_{E_p^{\gamma(\varepsilon)-\frac{1}{2}}} \\
&\qquad \leq c'''\|v-w\|_{L^{\frac{Np}{N-2p-4p\varepsilon}}(\mathbb{R}^N)} \left(\|v\|^{\rho-1}_{L^{\frac{N(\rho-1)}{4(1+\varepsilon-\gamma(\varepsilon))}}(\mathbb{R}^N)} + \|w\|^{\rho-1}_{L^{\frac{N(\rho-1)}{4(1+\varepsilon-\gamma(\varepsilon))}}(\mathbb{R}^N)}\right).
\end{aligned}
\tag{3.23}
$$

According to the first embedding in (3.20) the right hand side of (3.23) will be bounded by the right hand side of (3.18) provided that

$$
H_p^{2+4\varepsilon}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np}{N-2p-4p\varepsilon}}(\mathbb{R}^N) \tag{3.24}
$$

and

$$
H_p^{2+4\varepsilon}(\mathbb{R}^N) \hookrightarrow L^{\frac{N(\rho-1)}{4(1+\varepsilon-\gamma(\varepsilon))}}(\mathbb{R}^N), \tag{3.25}
$$

where ε is limited by the condition

$$
\varepsilon \in (0, \beta^*(p) - \frac{1}{2}]. \tag{3.26}
$$

Application of (3.20) and Hölder's inequality in (3.22)-(3.23) requires that

$$\frac{1}{2} \geq \gamma(\varepsilon) \geq \frac{1}{2} - \beta^*(p'), \quad p \geq \frac{Np}{N + (2 - 4\gamma(\varepsilon)p} > 1,$$
$$\frac{N-2p}{4p} > \varepsilon, \quad \theta = \frac{N + (2 - 4\gamma(\varepsilon))p}{N - 2p - 4p\varepsilon} > 1,$$

for which it is sufficient to assume (3.21) and

$$\frac{1}{2} \geq \gamma(\varepsilon) > \frac{2p - Np + N}{4p} =: \tilde{\gamma}, \qquad \frac{N-2p}{4p} > \varepsilon > 0. \tag{3.27}$$

Note that requirement that $\gamma(\varepsilon) \geq \frac{1}{2} - \beta^*(p')$ is not a real restriction here, since when $\beta^*(p') \geq \frac{1}{2}$, we have $\gamma(\varepsilon) > 0$, whereas when $\beta^*(p') < \frac{1}{2}$, from (3.27) we have $\tilde{\gamma} > \frac{1}{2} - \beta^*(p')$. Also note that, since $r > \frac{N}{4}$, the second condition in (3.27) above implies (3.26) and then (3.24) holds true as well.

On the other hand, for (3.25) one needs (3.26) and

$$\overline{\gamma} := \frac{(\rho-1)(2 + 4\varepsilon - \frac{N}{p}) + 4(1+\varepsilon)}{4} \geq \gamma(\varepsilon) \geq \frac{4p(1+\varepsilon) - N(\rho-1)}{4p} = \underline{\gamma}. \tag{3.28}$$

We remark that $\overline{\gamma} > \underline{\gamma}$ and $\overline{\gamma} \geq \varepsilon\rho$ for every $\rho \in (1, \frac{N+2p}{N-2p}]$, $\varepsilon > 0$. Furthermore, $\frac{1}{2} \geq \overline{\gamma}$ if $\frac{(N-2p)(\rho-1)}{4p\rho} - \frac{1}{2\rho} \geq \varepsilon$ and $\overline{\gamma} > \tilde{\gamma}$ for $\varepsilon > \max\{0, \frac{N(\rho-p)-2p\rho}{4p\rho}\}$, whereas via (3.17) we have that $\frac{(N-2p)(\rho-1)}{4p\rho} - \frac{1}{2\rho} > \max\{\frac{N(\rho-p)-2p\rho}{4p\rho}, 0\}$.

What was said above ensures that, the inequalities (3.21), (3.27) and (3.28) have nonempty set of solutions, which consists of triples $(\rho, \varepsilon, \gamma(\varepsilon))$, where

$$\rho \in (\frac{N}{N-2p}, \frac{N+2p}{N-2p}], \quad \varepsilon \in (\max\{0, \frac{N(\rho-p) - 2p\rho}{4p\rho}\}, \frac{(N-2p)(\rho-1)}{4p\rho} - \frac{1}{2\rho}]$$

and

$$\gamma(\varepsilon) \in [\rho\varepsilon, \overline{\gamma}] \cap [\underline{\gamma}, \overline{\gamma}] \cap (\tilde{\gamma}, \overline{\gamma}] =: \mathcal{I}(\varepsilon).$$

Observe that for any admissible $(\rho, \varepsilon, \gamma(\varepsilon))$ the left hand side inequality in (3.28) leads to the condition

$$\rho \leq \frac{N + 2p - 4p\gamma(\varepsilon)}{N - 2p - 4p\varepsilon}$$

and (3.21) also to

$$\rho \leq \frac{N + 2p - 4p\rho\varepsilon}{N - 2p - 4p\varepsilon}.$$

The latter inequality holds if and only if $\rho \leq 1 + \frac{4p}{N-2p} = \rho_c^2$ and the value $\rho = \rho_c^2$ cannot be attained for $\gamma(\varepsilon) > \rho_c^2\varepsilon$ but only for $\gamma(\varepsilon) = \rho_c^2\varepsilon$; that is, if $\rho = \rho_c^2$, $\mathcal{I}(\varepsilon) = \{\varepsilon\rho_c^2\}$.

Observe finally that (1.18) holds whenever $\rho < \rho_c^2$ as in this case $\gamma(\varepsilon) > \rho_c^2\varepsilon$ (see [3], also [6, Corollary 1.1]).

Hence (3.18) is satisfied and local well posedness of (1.1) follows from [3, Corollary 1]. In addition, we have from the results in [3, Corollary 1] that

$$u \in C([0, \tau_0), E_p^{\frac{1}{2}}) \cap C((0, \tau_0), E_p^{\frac{1}{2}+\gamma(\varepsilon)}) \cap C^1((0, \tau_0), E_p^{\gamma(\varepsilon)-\frac{1}{2}}). \tag{3.29}$$

Since the above analysis shows that $\gamma(\varepsilon)$ can be chosen equal $\frac{1}{2}$ then from (3.29) we obtain (1.16) as $E_p^1 \hookrightarrow H_p^{4\beta^*(p)}(\mathbb{R}^N)$. $\square$

Remark 3.3. *Due to the above consideration it suffices to assume that g belongs to $E_p^{\gamma(\varepsilon)-\frac{1}{2}}$ for arbitrarily fixed $\gamma(\varepsilon) \in \mathcal{I}(\varepsilon)$, $\varepsilon \in (\max\{0, \frac{N(\rho-p)-2p\rho}{4p\rho}\}, \frac{(N-2p)(\rho-1)}{4p\rho} - \frac{1}{2\rho}]$ and the problem (1.1) remains well posed in $H_p^2(\mathbb{R}^N)$ although the solution is less regular than it is stated in (1.16).*

Now we prove the following result, mentioned above.

Proposition 3.4. *Assume (1.2)-(1.4) and (1.11). Suppose also that $g \in L^p(\mathbb{R}^N)$ with a certain $p \in [2,\infty)$ and $2 < \frac{N}{p}$ and (1.5) holds with some $1 < \rho < 1 + \frac{4p}{N-2p} =: \rho_c^2$.*

Then the map $\mathcal{F}$ defined in (3.1) is Lipschitz continuous on bounded sets from $E_p^{\frac{1}{2}} = H_p^2(\mathbb{R}^N)$ into $E_p^{\gamma-\frac{1}{2}}$ with

$$\gamma := \min\{\frac{N+2p-\rho(N-2p)}{4p}, \frac{1}{2}\} \in (0, \frac{1}{2}].$$

Actually, whenever $N > 2p$, $1 < \rho \le \frac{N}{N-2p}$ and $p \in (1,\infty)$, $\mathcal{F}$ is Lipschitz continuous on bounded sets from $E_p^{\frac{1}{2}} = H_p^2(\mathbb{R}^N)$ into $E_p^0 = L^p(\mathbb{R}^N)$.

Proof: Note that $\gamma < \frac{1}{2}$ if and only if $\rho \in (\frac{N}{N-2p}, \frac{N+2p}{N-2p})$. In what follows we first prove the last part of the proposition.

If $\rho \in (1, \frac{N}{N-2p}]$ then for f_{02} defined in Lemma 3.1 we have that

$$\begin{aligned}
\|f_{02}(v) - f_{02}(w)\|_{L^p(\mathbb{R}^N)} &\le c\||v-w|(|v|^{\rho-1} + |w|^{\rho-1})\|_{L^p(\mathbb{R}^N)} \\
&\le c\|v-w\|_{L^{\frac{Np}{N-2p}}(\mathbb{R}^N)} \||v|^{\rho-1} + |w|^{\rho-1}\|_{L^{\frac{N}{2}}(\mathbb{R}^N)} \\
&\le c\|v-w\|_{L^{\frac{Np}{N-2p}}(\mathbb{R}^N)} \big(\|v\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2}}(\mathbb{R}^N)} + \|w\|^{\rho-1}_{L^{\frac{N(\rho-1)}{2}}(\mathbb{R}^N)}\big) \\
&\le c\|v-w\|_{H_p^2(\mathbb{R}^N)} \big(\|v\|^{\rho-1}_{H_p^2(\mathbb{R}^N)} + \|w\|^{\rho-1}_{H_p^2(\mathbb{R}^N)}\big).
\end{aligned}$$

Using Lemma 3.1 we obtain that in the above situation $\mathcal{F}$ is Lipschitz continuous on bounded sets from $E_p^{\frac{1}{2}} = H_p^2(\mathbb{R}^N)$ into $E_p^0 = L^p(\mathbb{R}^N)$.

Assume now that $\rho \in (\frac{N}{N-2p}, \frac{N+2p}{N-2p})$. Then, by assumption, $N > 2p > 4$ and $p \ge \frac{Np}{N+(2-4\gamma)p} = \frac{Np}{\rho(N-2p)} > 1$. Also note that $\frac{N(\rho-1)}{4(1-\gamma)} = \frac{Np}{N-2p} \ge p$. Using (3.19)-(3.20) and Hölder's inequality with $\theta = \frac{N+(2-4\gamma)p}{N-2p}$ and $\theta' = \frac{\theta}{\theta-1} = \frac{N+(2-4\gamma)p}{4(1-\gamma)p}$ we get

$$
\begin{aligned}
&\|f_{02}(v)-f_{02}(w)\|_{E_p^{\gamma-\frac{1}{2}}} \le c'\|f_{02}(v)-f_{02}(w)\|_{L^{\frac{Np}{N+(2-4\gamma)p}}(\mathbb{R}^N)}\\
&\le c'c\||v-w|(|v|^{\rho-1}+|w|^{\rho-1})\|_{L^{\frac{Np}{N+(2-4\gamma)p}}(\mathbb{R}^N)}\\
&\le c'c\|v-w\|_{L^{\frac{Np}{N-2p}}(\mathbb{R}^N)}\||v|^{\rho-1}+|w|^{\rho-1}\|_{L^{\frac{N}{4(1-\gamma)}}(\mathbb{R}^N)}\\
\le& c''\|v-w\|_{H^2_p(\mathbb{R}^N)}\Big(\|v\|^{\rho-1}_{L^{\frac{N(\rho-1)}{4(1-\gamma)}}(\mathbb{R}^N)}+\|w\|^{\rho-1}_{L^{\frac{N(\rho-1)}{4(1-\gamma)}}(\mathbb{R}^N)}\Big)\\
\le& c'''\|v-w\|_{E_p^{\frac{1}{2}}}\Big(\|v\|^{\rho-1}_{E_p^{\frac{1}{2}}}+\|w\|^{\rho-1}_{E_p^{\frac{1}{2}}}\Big),
\end{aligned}
$$

and using Lemma 3.1 we complete the proof. □

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