PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2010-06

Isothermalisation for a non-local heat equation

Emmanuel Chasseigne and Raúl Ferreira

Febrero-2010

http://www.mat.ucm.es/deptos/ma
e-mail:matemática_aplicada@mat. ucm.es

Isothermalisation for a Non-local Heat Equation

Emmanuel Chasseigne
Laboratoire de Mathématiques et Physique Théorique,
Universite de Tours, Parc de Grandmont, F37200 Tours, France
e-mail: echasseigne@univ-tours.fr

Raúl Ferreira
Departamento de Matemática Aplicada, Fac. de C.C. Qumicas,
U. Complutense de Madrid, 28040 Madrid, Spain
e-mail: raul_ferreira@ucm.es

February 19, 2010

Abstract

In this paper we study the asymptotic behavior for a nonlocal heat equation in an inhomogenous medium:

$$\rho(x)u_t = J * u - u \text{ in } \mathbb{R}^N \times (0,\infty)\,,$$

where ρ is a continous positive function, u is nonnegative and J is a probability measure having finite second-order momentum. Depending on integrability conditions on the initial data u_0 and ρ, we prove various isothermalisation results, *i.e.* $u(t)$ converges to a constant state in the whole space.

Keywords: Nonlocal diffusion, asymptotic behaviour, nonhomogeneous media.

MSC2010: 35B40, 35R09, 45A05, 45K05.

1 Introduction

The aim of this paper is to study the asymptotic behavior for a nonlocal heat equation in an inhomogenous medium:

$$\begin{cases} \rho(x)u_t = J * u - u, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1}$$

Here, u_0 is a nonnegative continuous function in $\mathbb{R}^N$ and $*$ denotes the convolution with a kernel $J : \mathbb{R}^N \to \mathbb{R}$, which is a radial, continuous probability density having finite second-order momentum:

$$\int_{\mathbb{R}^N} J(s)\,\mathrm{d}s = 1\,, \quad \mathbb{E}(J) = \int_{\mathbb{R}^N} sJ(s)\,\mathrm{d}s = 0\,,\ \mathbb{V}(J) = \int_{\mathbb{R}^N} s^2 J(s)\,\mathrm{d}s < +\infty\,.$$

Typical examples of kernels that we consider are the gaussian law, the exponential law or any compactly supported kernel. We also assume that ρ is a positive, continuous function in $\mathbb{R}^N$, whether integrable or not.

The operator $J * u - u$ can be interpreted as a non-local diffusion operator. Indeed, if $u(x,t)$ represents the density of a single poulation and $J(x-y)$ is the probability to jump from y to x then the term $(J*u)(x)$ is the rate at which individuals arrive to x and $-u(x)$ is the rate at which individuals leave from x, see for instance [7]. In the case of heat propagation, u stands for a temperature and $\rho(x)$ represents the density of the medium.

Problem (1) is called *non-local* because the diffusion at $u(x,t)$ depends on all the values of u in the support of J and not only of the value of $u(x,t)$, as it is the case for the local diffusion problem

$$\begin{cases} \rho(x)u_t = \Delta u, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

For this local problem it is well know that for dimension $N = 1, 2$ there exists a unique solution in the class of bounded solutions, see [8] and [5]. Moreover, if $\rho \in \mathrm{L}^1(\mathbb{R}^N)$ and u_0 is bounded, then as $t \to \infty$ the solution converges on compact sets to $\mathbb{E}_\rho(u_0)$, the mean of u_0 with respect to ρ :

$$\mathbb{E}_\rho(u_0) := \frac{\displaystyle\int_{\mathbb{R}^N} u_0(y)\rho(y)\,\mathrm{d}y}{\displaystyle\int_{\mathbb{R}^N} \rho(y)\,\mathrm{d}y} \in \mathbb{R}_+ \,.$$

This phenomenon is called *isothermalisation*, since the heat distribution converges to a non-trivial isothermal state in all the space. However, for dimension $N \geqslant 3$ uniqueness is lost in the class of bounded solutions and some solutions decrease to zero as $t \to \infty$, so that isothermalisation does not take place, see [9] and [6].

On the contrary here, we show that if ρ is integrable, given any bounded initial data there exists a unique classical solution of problem (1) and that the isothermalisation effect always occurs for all dimension $N \geqslant 1$ (see more precise statements below).

The case when ρ is not integrable is also considered, which is more related to the study of the homogeneous case ($\rho \equiv 1$), see [3] and [1]. For bounded solutions, the flux at infinity is so big that solutions go down to zero asymptotically while if the data is unbounded, the solution may go to infinity asymptotically as $t \to \infty$.

Organisation and main results

We first prove in Section 2 a comparison result which gives uniqueness for problem (1). In Section 3 we study the existence in the class of bounded solutions, which requires an approximation first by functions ρ_n that do not degenerate at infinity. The main theorem is the following:

Theorem 1.1 *Let $\rho > 0$, continous, and u_0 be a bounded nonnegative continuous function. Then there exits a unique classical solution of problem* (1).

Before studying the asymptotic behaviour, an important step in this direction consists first in proving that if ρ is integrable and u_0 bounded, the following conservation law holds:

$$\int_{\mathbb{R}^N} \rho(x)u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x\,.$$

This is a nontrivial result which is obtained by approximation with Neuman problems in bounded domains, see Section 4. Then, Section 5 is devoted to study the isothermalisation phenomenon for bounded initial data. Then the main theorem is as follows:

Theorem 1.2 *Let $\rho > 0$, continuous, integrable and u_0 be a bounded nonnegative continuous function. Then $u(x,t) \to \mathbb{E}_\rho(u_0)$ as $t \to \infty$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N)$ for any $1 \leqslant p < \infty$; the convergence also holds in $\mathrm{L}^1(\rho)$:*

$$\lim_{t\to\infty} \int_{\mathbb{R}^N} |u(x,t) - \mathbb{E}_\rho(u_0)|\,\rho(x)\,\mathrm{d}x = 0\,.$$

Notice that we have no restriction on the space dimension contrary to what happens for the "local" heat equation. In the case when ρ is not integrable but still $u_0 \in \mathrm{L}^1(\rho)$ (that is, $\rho u_0 \in \mathrm{L}^1(\mathbb{R}^N)$) the flux at infinity forces the solution to go to zero:

Theorem 1.3 *Let $\rho > 0$ and u_0 be a bounded nonnegative continuous function such that $u_0 \in \mathrm{L}^1(\rho)$. If ρ is not integrable in $\mathbb{R}^N$, then $u(x,t) \to 0$ as $t \to \infty$ in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N)$ for any $1 \leqslant p < \infty$.*

Finally, in Section 6 we investigate the case when the initial data is not bounded. We first prove an existence result in the class of unbounded solutions provided ρ does not degenerate too rapidly at infinity. More precisely, if

$$\rho(x) \geqslant \frac{\eta}{1+|x|^\gamma} \qquad \gamma \leqslant 2\,, \tag{2}$$

then we have an existence result for quadratic initial data:

Theorem 1.4 *Let u_0 be a positive continuous function with at most quadratic growth at infinity. If the function ρ satisfies* (2) *then there exists a minimal solution of problem* (1).

More generally, if ρ is integrable, we prove similar isothermalisation results for the minimal solution:

Theorem 1.5 *We assume that ρ is a continuous positive integrable function in $\mathbb{R}^N$ and that u_0 is a continuous nonnegative function, possibly unbounded, such that there exists a solution u. Noting $\underline{u}$ the minimal solution, the following holds:*

i) If $u_0 \in \mathrm{L}^1(\rho)$ the isothermalisation takes place in $\mathrm{L}^1(\rho)$,

$$\lim_{t\to\infty} \int_{\mathbb{R}^N} |\underline{u}(x,t) - \mathbb{E}_\rho(u_0)|\,\rho(x)\,\mathrm{d}x = 0\,.$$

ii) If $u_0 \notin \mathrm{L}^1(\rho)$, we have that for all $1 \leqslant p < \infty$,

$$\lim_{t\to\infty} \underline{u}(x,t) = \infty \quad in \quad \mathrm{L}^p_{loc}(\mathbb{R}^N).$$

In the case of nonintegrable ρ's with $u_0 \notin \mathrm{L}^1(\rho)$ the asymptotic behavior is more difficult to treat. For instance, if $\rho \equiv 1$, the solutions

$$u(x,t) = |x|^2 + \mathbb{V}(J)t, \qquad \text{and} \qquad u(x,t) = 1$$

have different behavior. Thus, there is a balance between ρ, J, and the initial data u_0 which is not easy to handle and the question remains open.

2 Preliminaries

Let us specify first what is the notion of solution that we use:

Definition 2.1 *Let $u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$. By a stong solution of* (1) *we mean a function $u \in \mathrm{C}^0\big([0,\infty);\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)\big)$ such that $u_t, J*u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,\infty)\big)$, the equation is satisfied in the $\mathrm{L}^1_{\mathrm{loc}}$-sense and such that $u(x,0) = u_0(x)$ almost everywhere in $\mathbb{R}^N$.*

We shall consider also solutions with more regularity:

Definition 2.2 *A classical solution of* (1) *is a solution such that moreover $u, u_t, J*u \in \mathrm{C}^0(\mathbb{R}^N \times [0,\infty))$ and the equation holds in the classical sense everywhere in $\mathbb{R}^N \times [0,\infty)$.*

A classical sub or supersolution is defined as usual with inequalities instead of equalities in the equation.

Now let us state a simple regularity Lemma, which contains a technical trick that we shall use several times in the sequel:

Lemma 2.1 *Let u be a strong solution of* (1)*. We assume moreover that u_0 is continuous in $\mathbb{R}^N$ and that the convolution term $J*u$ is continous in $\mathbb{R}^N \times [0,\infty)$. Then u and u_t are also continuous in $\mathbb{R}^N \times [0,\infty)$ and u is a classical solution.*

Proof. We introduce the following transform:

$$\mathcal{T}_\rho[u](x,t) := \mathrm{e}^{\,t/\rho(x)} u(x,t)\,. \tag{3}$$

A straightforward calculus show that $v = \mathcal{T}_\rho[u]$ satisfies

$$v_t = \frac{\mathrm{e}^{\,t/\rho(x)}}{\rho(x)}(J*u)(x,t)\,,$$

which is a continuous function in $\mathbb{R}^N \times [0,\infty)$. Integrating between 0 and t we get:

$$v(x,t) = \int_0^t \partial_t v(x,s)\,\mathrm{d}s + v(x,0) = \int_0^t \partial_t v(x,s)\,\mathrm{d}s + u(x,0)\,,$$

hence v is continuous in $\mathbb{R}^N \times [0,\infty)$. This implies that u is also continuous in $\mathbb{R}^N \times [0,\infty)$, and the equation holds in the classical sense. □

Remark 2.1 It is well-known in the convolution theory that under one of the following assumptions, the convolution term is continuous:
(i) u is bounded (since J is integrable);
(ii) J compactly supported and u locally integrable.

The following lemma concerns the comparison of classical sub/supersolutions of the problem.

Lemma 2.2 *Let $\overline{u}$ and $\underline{u}$ be continous functions in $\mathbb{R}^N \times [0,\infty)$. We assume that $\overline{u}$ is a classical supersolution of* (1) *and that $\underline{u}$ is a bounded classical subsolution of* (1) *with $\overline{u}(x,0) \geqslant \underline{u}(x,0)$. Then $\overline{u} \geqslant \underline{u}$ in $\mathbb{R}^N \times \mathbb{R}_+$.*

Proof. We consider the functions

$$w_\delta = \underline{u} - \overline{u} - \delta - \frac{\delta}{4\mathbb{V}(J)}\,|x|^2\,, \quad \text{and} \quad \phi_\delta = \mathcal{T}_\rho[w_\delta]\,.$$

Since

$$J * |x|^2 - |x|^2 = \int_{\mathbb{R}^N} J(y)|y|^2\,\mathrm{d}y - 2\int_{\mathbb{R}^N} J(y) < x, y > \,\mathrm{d}y = \mathbb{V}(J)$$

the above functions are related by the inequality

$$\rho(x)(\phi_\delta)_t = \mathrm{e}^{\,t/\rho(x)}\Big((J * w_\delta)(x,t) + \frac{\delta}{4}\Big)\,. \tag{4}$$

The function w_δ is continuous in $\mathbb{R}^N \times [0,\infty)$ and notice that $w_\delta \leqslant -\delta < 0$ at time $t = 0$. There are two options:
(i) either $w_\delta \leqslant -\delta/2$ in $\mathbb{R}^N \times (0,\infty)$;
(ii) or there exists a point $(x,t) \in \mathbb{R}^N \times (0,\infty)$ such that $w_\delta(x,t) > -\delta/2$.

Let us assume that we are in this second case and let t_0 be defined as follows:

$$t_0 := \inf\{t > 0 : \exists x \in \mathbb{R}^N\,,\ w_\delta(x,t) > -\delta/2\} < \infty\,.$$

A first estimate shows that

$$w_\delta(x,t) > -\delta/2 \Longrightarrow |x| < R_\delta := \Big(\frac{4\mathbb{V}(J)\|\underline{u}\|_\infty}{\delta}\Big)^{1/2} < \infty\,,$$

so that the only possible points where w_δ may reach a level above $-\delta/2$ are located inside the fixed ball B_{R_δ}. Thus, up to a finite number of terms, any minimizing sequence remains inside the compact set $\overline{B}_{R_\delta} \times [0, t_0+1]$. After extraction and using the continuity of w_δ, we get that the inf is attained and there is a point (x_0,t_0) such that $w_\delta(x_0,t_0) = -\delta/2$. Of course this implies that t_0 cannot be zero, since $w_\delta \leqslant -\delta$ at $t = 0$.

So, $t_0 > 0$ and necessarily $w_\delta \leqslant -\delta/2$ in $\mathbb{R}^N \times (0,t_0)$, which implies also $J * w_\delta \leqslant -\delta/2$ for $t \in (0,t_0)$, everywhere in $\mathbb{R}^N$. Using (4), we see that $(\phi_\delta)_t \leqslant 0$ in $(0,t_0)$ so that

$$\forall (x,t) \in \mathbb{R}^N \times (0,t_0)\,,\ \phi_\delta(x,t) \leqslant \phi_\delta(x,0) = w_\delta(x,0) \leqslant -\delta\,.$$

But taking $x = x_0$ and letting $t \to t_0$, by continuity we arrive at $w_\delta(x_0,t_0) \leqslant \phi_\delta(x_0,t_0) \leqslant -\delta$ which is a contradiction.

So, we end up with the first possibility (i): for any $\delta > 0$, $w_\delta \leqslant -\delta/2$ in all $\mathbb{R}^N \times (0,\infty)$ and taking limit as $\delta \to 0$ we obtain

$$(\underline{u} - \overline{u})(x,t) \leqslant 0 \quad \text{for all} \quad x \in \mathbb{R}^N\,,\ t \geqslant 0\,,$$

which ends the proof. □

Remark 2.2 Observe that the same result holds if we only assume that the subsolution grows strictly less than $|x|^2$: indeed, if $\underline{u}(x,t) \leqslant C(1+|x|^{2-\varepsilon})$ for some $\varepsilon \in (0,2)$ then the point x_0 remains inside a fixed ball with radius $O(\delta^{-1/\varepsilon})$ and the rest of the proof follows identically. See also [1].

An immediate consequence of the comparison property is the uniqueness of the bounded classical solution.

Corollary 2.1 *Given u_0 continous and bounded in $\mathbb{R}^N$, there exists at most one classical solution u of* (1).

Proof. Since u_0 is bounded, then the constant $\|u_0\|_\infty$ is a bounded (classical) supersolution. Hence, if there exists a (classical) solution u, using the comparison result we see that u is necessarily bounded by $\|u_0\|_\infty$. Then if we have two solutions u_1 and u_2, since both are bounded we can again use the comparison result in both ways to get that $u \leqslant v$ and $v \leqslant u$ so that the solution is unique. □

3 Existence and uniqueness of bounded solutions

We shall assume first that ρ is bounded below by some positive constant, so that we can apply the usual fixed-point approach to prove the existence of the solution for integrable initial data. Then we derive a general result for possibly degenerating ρ's by approximation.

3.1 Non-degenerate case.

Lemma 3.1 *Let $\rho \geqslant \rho_0 > 0$, continuous and $u_0 \in \mathrm{L}^1(\mathbb{R})$. Then there exists a unique strong solution u of* (1) *in the class $\mathrm{C}^0\big([0,\infty);\mathrm{L}^1(\mathbb{R}^N)\big)$.*

Proof. Following [3], we consider $t_0 > 0$ to be fixed later on and the space $X_{t_0} = \mathrm{C}^0([0,t_0];\mathrm{L}^1(\mathbb{R}^N))$ equipped with the norm $|||w||| = \max\big\{\|w(\cdot,t)\|_{\mathrm{L}^1}, 0 \leqslant t \leqslant t_0\big\}$. Then we define an operator $T : X_{t_0} \to X_{t_0}$ as follows:

$$T_{w_0}(w)(x,t) := w_0(x) + \frac{1}{\rho_n}\int_0^t \Big\{(J * w)(x,s) - w(x,s)\Big\}\,\mathrm{d}s$$

A straightforward calculus, using that $\rho \geqslant \rho_0$ shows that:

$$|||T_{w_0}(w) - T_{z_0}(z)||| \leqslant \|w_0 - z_0\|_{\mathrm{L}^1} + \frac{2t_0}{\rho_0}|||w - z|||\,.$$

We deduce that for any fixed initial data $u_0 \in \mathrm{L}^1(\mathbb{R}^N)$, if t_0 is sufficiently small, T_{u_0} is a contraction, hence there exists a unique solution u starting with $u(0) = u_0$, defined up to $t = t_0$:

$$u(x,t) = u_0(x) + \frac{1}{\rho}\int_0^t \Big\{(J * u)(x,s) - u(x,s)\Big\}\,\mathrm{d}s\,,$$

which implies that $\partial_t u$ exists as a $\mathrm{L}^1_{\mathrm{loc}}$ function, and that (1) holds in $(0, t_0)$ in the strong sense (not classical though, a priori). We then iterate the argument to construct a solution for all time $t > 0$. □

3.2 Degenerate case

We construct here a solution of (1) for continuous and bounded initial data, using the approximation $\rho_n \to \rho$ and also an approximation of initial data from $\mathrm{L}^1(\mathbb{R}^N)$. We consider the following problem:

$$\begin{cases} \rho_n(x)(u_n)_t = J * u_n - u_n, & (x,t) \in \mathbb{R}^N \times (0,\infty), \\ u_n(x,0) = u_0(x)\chi_n(x), & x \in \mathbb{R}^N, \end{cases} \tag{5}$$

where by χ_n we denote the indicator function of B_n (the ball of radius n centered at the origin) and $\rho_n(x) = \max\{\rho(x); \alpha_n\}$ where (α_n) is a sequence of positive real numbers strictly decreasing to zero. With this choice, $\rho_n \geqslant \alpha_n$, ρ_n decreases to ρ, and $\rho_n = \rho$ in any compact set for n sufficiently big (since ρ is continuous and positive).

Since $\rho_n \geqslant \alpha_n > 0$ and $u_0\chi_n \in \mathrm{L}^1(\mathbb{R}^N) \cap \mathrm{L}^1(\rho_n)$ we can apply the results of the previous Subsection to obtain that there exist a classical solution u_n.

Now, in order to obtain a solution of problem (1) we pass to the limit as $n \to \infty$.

Proposition 3.1 *Let $\rho > 0$, continuous and $u_0 \in \mathrm{L}^\infty(\mathbb{R}^N) \cap \mathrm{C}^0(\mathbb{R}^N)$ be a nonnegative function. For any positive decreasing sequence $\alpha_n \searrow 0$, let u_n be the solution of (5) associated to $\rho_n = \max\{\rho, \alpha_n\}$. Then the sequence (u_n) converges in $\mathrm{L}^1_{\mathrm{loc}}$ to the unique classical solution $u \in \mathrm{C}^0\big(\mathbb{R}^N \times [0,\infty)\big)$ of (1) with initial data u_0.*

Proof. Since $\rho_n \geqslant \alpha_n > 0$ does not degenerate, we may use Lemma 3.1 to get a solution $u_n \in \mathrm{C}^0\big([0,\infty); \mathrm{L}^1(\mathbb{R}^N)\big)$ with ρ_n. We first observe that $\|u_0\|_\infty$ is a bounded classical supersolution of (5) with $0 \leqslant u_{0n} \leqslant \|u_0\|_\infty$, so we may apply Lemma 2.2 with $\rho = \rho_n$ to get that

$$0 \leqslant u_n(x,t) \leqslant \|u_0\|_\infty\,.$$

This allows to pass to the limit in L^∞-weak* along a subsequence (still denoted by u_n): there exists a bounded function u such that $u_n \rightharpoonup u$.

We then deduce that $J * u_n \to J * u$ pointwise (since J is integrable), and $J * u$ is continuous since u is bounded (see Remark 2.1). Moreover, $J * u_n$ is uniformly bounded so that the convergence also holds in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$. On the other hand, $\rho_n \partial_t u_n$ converges weakly in the sense of distributions to $\rho\partial_t u$, which would be enough to obtain a weak (distributional) solution. But we have more.

Using transform $\mathcal{T}_{\rho_n}$ introduced in Lemma 2.1, we see that $v_n := \mathcal{T}_{\rho_n}[u_n]$ satisfies the equation:

$$\partial_t v_n(x,t) = \frac{\mathrm{e}^{\,t/\rho_n(x)}}{\rho_n(x)}(J * u_n)(x,t)\,,$$

hence $\partial_t v_n$ converges pointwise and in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$ to some f which is continuous. Integrating in time we get:

$$v_n(x,t) = u_{0n}(x) + \int_0^t \partial_t v_n(x,s)\,\mathrm{d}s \to v(x,t) = u_0(x) + \int_0^t f(x,s)\,\mathrm{d}s\,,$$

which is continuous in $\mathbb{R}^N \times [0,\infty)$.

Hence $u_n = \mathrm{e}^{\,-t/\rho_n(x)} v_n$ also converges pointwise and in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$, and the limit is necessarily u, which is a continuous function in $\mathbb{R}^N \times [0,\infty)$. Coming back to the equation for u_n, we see that every term converges strongly, and in the limit we have:

$$\rho\, u_t = J * u - u\,,$$

where all the terms of the equation are in fact in $\mathrm{C}^0(\mathbb{R}^N \times [0,\infty))$. Hence u is a classical solution of (1) and Corollary 2.1 implies its uniqueness. The last step of the existence proof consists in saying that since any possible limit is in fact unique, then all the sequence converges to u, and there is no need to extract. □

Proof of Theorem 1.1. It is just the combination of Corollary 2.1 and Proposition 3.1. □

3.3 Approximation by Neumann problems

Another way to get a solution of (1) in $\mathbb{R}^N$ consists in solving first the equation in the balls B_n and pass to the limit as $n \to \infty$. Since we are interested in getting a conservation law, it is natural to consider here the following Neuman problem, where χ_n denotes the indicator of B_n:

$$\begin{cases} \rho(x)\partial_t u_n = \displaystyle\int_{B_n} \big(u_n(y,t) - u_n(x,t)\big) J(x-y)\,\mathrm{d}y\,, & x \in B_n\,, \\ u_n(x,0) = u_0\chi_n(x)\,, \end{cases} \tag{6}$$

Lemma 3.2 *Let $\rho > 0$ be a continuous function in $\mathbb{R}^N$ and u_0 be a nonnegative continuous function such that $u_0 \in \mathrm{L}^1(\rho) \cap \mathrm{L}^\infty$. Then for any $n \geqslant 0$, there exists a unique solution $u_n \in \mathrm{C}^0(0,T;\mathrm{L}^1(B_n))$ of (6), and u_n is continuous. Moreover, the maximum principle holds which implies $0 \leqslant u_n \leqslant \|u_0\|_\infty$ and the conservation law holds:*

$$\int_{B_n} u_n(x,t)\rho(x)\,\mathrm{d}x = \int_{B_n} u_0(x)\rho(x)\,\mathrm{d}x\,.$$

Proof. Existence and uniqueness is done in [3] using a semi-group approach, in the case $\rho \equiv 1$. Even if ρ is not constant here, it is bounded from below in B_n so that the modifications are similar to those of the non-degenerate case, section 3.1. Moreover, using the same techniques as in [2], the equation (or inequation for sub/super solutions) still holds at the boundary, $\partial B_n \times (0,\infty)$. The comparison principle is obtained in the same way as in lemma 2.2, without the term $|x|^2$. Therefore, using the constant solutions $c = 0$ and $c = \|u_0\|_\infty$ to compare with the solution u_n, we get the desired estimate. Finally, the conservation law is simply obtained by integrating the equation over $B_n \times [0,t]$. □

Proposition 3.2 *Let $\rho > 0$ be a continuous function in $\mathbb{R}^N$ and u_0 be a nonnegative continuous function such that $u_0 \in \mathrm{L}^1(\rho) \cap \mathrm{L}^\infty$ and for any integer $n > 0$, let u_n be the solution constructed in Lemma 3.2. Then as $n \to \infty$, along a subsequence $u_n \to u$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$ where u is the unique classical solution of* (1).

Proof. Since the sequence u_n is bounded, it converges (along a subsequence still denoted u_n) in L^∞-weak$*$ to some nonnegative and bounded function u. Since $J \in \mathrm{L}^1$, this implies that $J * u_n \to J * u$ strongly and that $\partial_t u_n \to \partial_t u$ in the sense of distributions. We can then pass to the limit in the sense of distributions but here also we want a better convergence.

Let us introduce a modified version of transform $\mathcal{T}$ as follows:

$$v_n(x,t) := \mathrm{e}^{\,t(J*\chi_n)(x)/\rho(x)} w(x,t)\,.$$

Since (6) can be written as

$$\rho(x)\partial_t(u_n)(x,t) = \big[J*(u_n\chi_n)\big](x,t) - (J*\chi_n)(x)u_n(x)\,,$$

it follows immediately that v_n satisfies the equation

$$\partial_t v_n = \frac{\mathrm{e}^{\,t(J*\chi_n)(x)/\rho(x)}}{\rho(x)} J * u_n\,.$$

This implies that $\partial_t v_n$ converges strongly on compact sets of $\mathbb{R}^N \times [0,\infty)$, and so does $v_n(x,t) = u_0(x)\chi_n(x) + \int_0^t (v_n)_t(x,s)\,\mathrm{d}s$. Then u_n also converges strongly on compact sets of $\mathbb{R}^N \times [0,\infty)$ to its limit u.

Passing to the limit in the equation, we see that u is a strong solution of (1), which implies that it is the unique classical solution of this equation – this follows from Lemma 2.1 and Corollary 2.1. Any other converging subsequence leads to the same solution u so that all the sequence u_n converges to u in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N \times [0,\infty))$. □

Remark 3.1 *By uniqueness of classical solutions, we have that the classical solution obtained by approximation of the function $\rho(x)$ and the classical solution obtained as the limit of solutions of Neumann problems are identical.*

4 Conservation Law

In this Section we investigate various conditions under which solutions of (1) satisfy a conservation law. A first lemma in this direction is as follows:

Lemma 4.1 *Let ρ be a nonnegative measurable function and u a strong solution of* (1) *such that $u \in \mathrm{C}^0\big([0,\infty);\mathrm{L}^1(\mathbb{R}^N)\big)$. If $u_0(x) \in \mathrm{L}^1(\rho)$ then for any $t>0$, $u(\cdot,t) \in \mathrm{L}^1(\rho)$ and*

$$\int_{\mathbb{R}^N} \rho(x)u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x.$$

Proof. First, we integrate equation (1) with respect to the time variable to obtain

$$\rho(x)u(x,t) = \rho(x)u_0(x) + \int_0^t (J*u-u)(x,t)\,\mathrm{d}t.$$

We integrate over B_R using Fubini's theorem:

$$\int_{B_R} \rho(x)u(x,t)\,\mathrm{d}x + \int_0^t\int_{B_R} u(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_{B_R} \rho(x)u_0(x)\,\mathrm{d}x + \int_0^t\int_{B_R} J*u(x,t)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Using the fact that $u_0(x) \in \mathrm{L}^1(\rho)$ and $u \in \mathrm{C}^0\big([0,\infty);\mathrm{L}^1(\mathbb{R}^N)\big)$, we can use monotone convergence for the right-hand side terms which implies that $u(\cdot,t) \in \mathrm{L}^1(\rho)$. Moreover, since $u(\cdot,t) \in \mathrm{L}^1$,

$$\int_{\mathbb{R}^N} (J*u)(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} u(x,t)\,\mathrm{d}x, \quad \text{for any } t>0$$

so that finally we get

$$\int_{\mathbb{R}^N} \rho(x)u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x\,.$$

□

In particular, this result is valid for the solutions constructed in Section 3.1, but it is difficult in general to be sure that $u(t)$ remains in L^1. So we use the approximation by Neuman problems in order the keep the conservation law in the limit when $\rho \in \mathrm{L}^1$. If ρ is not integrable, we have only a one-sided estimate but this will be sufficient to obtain the asymptotic behaviour.

Proposition 4.1 *Let $\rho > 0$, continuous and $u_0 \in \mathrm{C}^0(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N) \cap \mathrm{L}^1(\rho)$. Then*
(i) for any $t>0$, $u(\cdot,t) \in \mathrm{L}^1(\rho)$ and

$$\int_{\mathbb{R}^N} \rho(x)u(x,t)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x\,.$$

(ii) if moreover we assume that $\rho \in \mathrm{L}^1(\mathbb{R}^N)$ then the conservation law holds:

$$\int_{\mathbb{R}^N} \rho(x)u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x\,.$$

Proof. Since $u \in \mathrm{L}^\infty \cap \mathrm{C}^0$, we may use the approximating sequence $\{u_n\}$ of Section 3.3, which satisfies that $u_n \to u$ at least almost everywhere, with $0 \leqslant u_n \leqslant \|u_0\|$ and

$$\int_{\mathbb{R}^N} u_n(x,t)\chi_n(x)\rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_0(x)\chi_n(x)\rho(x)\,\mathrm{d}x\,. \tag{7}$$

Since $u_0 \in \mathrm{L}^1(\rho)$, the dominated convergence theorem yields the convergence of the right-hand side integral as $n \to \infty$. For the left hand side, using Fatou's Lemma we obtain that

$$\int_{\mathbb{R}^N} \rho(x)u(x,t)\,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^N} \rho(x)u_n(x,t)\chi_n(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x\,,$$

which proves assumption (i) of the proposition.

Finally, if we assume that $\rho \in \mathrm{L}^1(\mathbb{R}^N)$ we can use also the dominated convergence theorem for the sequence $u_n\chi_n\rho$, which is bounded by $\|u_0\|_\infty\,\rho \in \mathrm{L}^1$. We then pass to the limit at the left-hand side of (7) and get (ii). □

5 Asymptotic behaviour for bounded solutions

We shall now derive our main results concerning the asymptotic behaviour for (1). We divide the proof in several steps.

5.1 Weak limit

This first step is easy, it only comes from the fact that the solution is globally bounded:

Lemma 5.1 *Let $\rho > 0$, continuous and $u_0 \in \mathrm{C}^0(\mathbb{R}^N)\cap\mathrm{L}^\infty(\mathbb{R}^N)$. Let u be the associated classical solution. Then for any $s > 0$ there exists a subsequence $t_k \to +\infty$ such that the following limit exists in L^∞-weak*:*

$$u_\infty(x,s) := \lim_{t_k\to\infty} u(x,s+t_k)\,.$$

Proof. Since u_0 is bounded, u is also bounded by comparison in the class of classical solutions, thus there exists a subsequence $t_k \to \infty$ such that $u(\cdot,s+t_k)$ converges in L^∞-weak* to a function $u_\infty(\cdot,s) \in \mathrm{L}^\infty(\mathbb{R}^N)$. □

5.2 Lyapounov functional

We now want a stronger result, so we use a Lyapounov functional:

Lemma 5.2 *Assume the hypotheses of Lemma 5.1 and that $u_0 \in \mathrm{L}^2(\rho)$. Then there exists a constant $C = C(u_0,\rho)$ such that,*

$$\int_t^\infty \int_{\mathbb{R}^N} \rho(x)(u_t)^2(x,s)\,\mathrm{d}s \leqslant C.$$

Proof. First we prove that for the approximating problem (5) the following functional

$$F[u_n](t) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u_n(x,t)-u_n(y,t))^2\,\mathrm{d}x\,\mathrm{d}y$$

is a Lyapunov functional which is nonincreasing along the evolution orbits. Multiply equation (5) by $(u_n)_t$ and integrating in space we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}F[u_n](t) = -4\int_{\mathbb{R}^N} \rho_n(x)((u_n)_t)^2(x,t)\,\mathrm{d}x\,, \tag{8}$$

while multiplying the equation by u_n gives

$$F[u_n](t) = -2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} \rho_n(x)u_n^2(x,t)\,\mathrm{d}x\,. \tag{9}$$

Notice that derivation under the integral is possible at the level n since we know that $u_n(t) \in \mathrm{C}^0\big(0,\infty;\mathrm{L}^1(\mathbb{R}^N)\big)$, which implies that also $\partial_t u_n \in \mathrm{C}(0,\infty;\mathrm{L}^1)$.

Integrating (9) we have that for some $C' = C'(u_0,\rho)$,

$$\int_0^t F[u_n](s)\,\mathrm{d}s = 2\int_{\mathbb{R}^N} \rho_n(x)u_0^2(x)\chi_n(x)\,\mathrm{d}x - 2\int_{\mathbb{R}^N} \rho_n(x)u_n^2(x,t)\,\mathrm{d}x \leqslant C'\,.$$

Indeed, as $u_0 \in \mathrm{L}^2(\rho)$ we have by monotone convergence that

$$\int_{\mathbb{R}^N} \rho_n(x)u_0^2(x)\chi_n(x)\,\mathrm{d}x \to \int_{\mathbb{R}^N} \rho(x)u_0^2(x)\,\mathrm{d}x < \infty\,.$$

Hence, $t \mapsto F[u_n](t)$ is a decreasing function which is in $\mathrm{L}^1(0,t)$ for all $t>0$. It follows that $F[u_n](t)$ must be bounded for all $t>0$ by some constant $C = C(u_0,\rho)$.

Moreover, $F[u_n](t)$ is positive so that, integrating (8), we get for any $t>0$:

$$\int_t^\infty \int_{\mathbb{R}^N} \rho((u_n)_t)^2(x,s)\,\mathrm{d}x\,\mathrm{d}s \leqslant \frac{1}{4}\,F[u_n](t) \leqslant C(u_0,\rho).$$

Using Fatou's Lemma and the fact that $\rho_n(u_n)_t$ converges strongly to ρu_t, we obtain the desired result. □

As and immediate consequence of this result we obtain:

Lemma 5.3 *Assume the hypotheses of Lemma 5.1 and that $u_0 \in \mathrm{L}^2(\rho)$. For all sequence $t_k \to \infty$ and $s>0$,*

$$\|\sqrt{\rho(\cdot)}\,u(\cdot,s+t_k) - \sqrt{\rho(\cdot)}\,u(\cdot,t_k)\|^2_{\mathrm{L}^2(\mathbb{R}^N)} \to 0 \quad \text{as } n\to\infty\,.$$

Hence, the limit function $u_\infty(x,s)$ does not depend on the variable $s>0$.

Proof. Note that for all sequence $t_k \to \infty$, we get

$$\begin{aligned}\big\|\sqrt{\rho(\cdot)}\,u(\cdot,s+t_k) - \sqrt{\rho(\cdot)}\,u(\cdot,t_k)\big\|^2_{\mathrm{L}^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \rho(x)\Big(\int_{t_k}^{t_k+s} u_t(x,\sigma)\,\mathrm{d}\sigma\Big)^2\,\mathrm{d}x \\ &\leqslant s\int_{\mathbb{R}^N}\int_{t_k}^{t_k+s} \rho(x)(u_t)^2(x,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}x \to 0.\end{aligned}$$

□

5.3 The $\omega-$limit set

We define the $\omega-$limit set as follows

$$\omega(u_0) = \{u_\infty \in \mathrm{C}^0(\mathbb{R}^N) \,:\, \exists t_j \to \infty \text{ such that } u(\cdot, t_j) \to u_\infty(\cdot) \text{ in } \mathrm{L}^\infty\text{-weak*}\}$$

Lemma 5.4 *Under the hypotheses of Lemma 5.1 and that $u_0 \in \mathrm{L}^2(\rho)$, the ω-limit set is reduced to constants.*

Proof. Since $u(x, s+t_k)$ converges weakly in L^∞-weak*, then as $t_k \to \infty$,

$$(J*u)(x, s+t_k) = \int J(x-y)u(y, s+t_k)\,\mathrm{d}y \overset{\text{pointwise}}{\longrightarrow} (J*u_\infty)(x,s)\,.$$

Moreover, since u is bounded, the convergence of $J*u$ is also strong in $\mathrm{L}^1_{\mathrm{loc}}$. On the other hand, $\partial_t \rho u(s+t_k) \to \rho\partial_s u_\infty(s)$ in the sense of distributions. We then pass to the limit in the sense of distributions in the equation and get

$$\rho(x)\frac{\partial}{\partial s}u_\infty(x,s) = J*u_\infty(x,s) - u_\infty(x,s)\,.$$

Using Lemma 5.3 we know that u_∞ is independent of s so that u_∞ is a bounded solution (in the sense of distributions) of

$$J*u_\infty - u_\infty = 0 \ \text{ in } \ \mathbb{R}^N\,.$$

We deduce that u_∞ is continuous because the convolution term is continous, and so it is necessarily constant in all $\mathbb{R}^N$. Indeed, the equation for u_∞ implies that it is harmonic (see for instance [4] for a proof) and bounded harmonic functions are constant. $\square$

5.4 Identification of the limit

We are now ready to identify the ω-limit set.

Lemma 5.5 *We assume the hypotheses of Lemma 5.1 and that $u_0 \in \mathrm{L}^1(\rho)$. Then the following holds:*

(i) if $\rho \in \mathrm{L}^1(\mathbb{R}^N)$, $\omega(u_0) = \big\{\mathbb{E}_\rho(u_0)\big\}$;

(ii) if $\rho \notin \mathrm{L}^1(\mathbb{R}^N)$, $\omega(u_0) = \big\{0\big\}$.

Proof. Notice first that since $u_0 \in \mathrm{L}^1(\rho)\cap\mathrm{L}^\infty$, then $u_0 \in \mathrm{L}^2(\rho)$, hence we may use Lemma 5.4. In the integrable case, $\rho \in \mathrm{L}^1$, we observe that as u is uniformly bounded, the dominated convergence Theorem gives

$$\int_{\mathbb{R}^N}\rho(x)u_n(x, s+t_j)\,\mathrm{d}x \to u_\infty\int_{\mathbb{R}^N}\rho(x)\,\mathrm{d}x\,.$$

Therefore, by Proposition 4.1-(*ii*) we obtain $u_\infty = \mathbb{E}_\rho(u_0)$, so that the ω-limit set is reduced to $\big\{\mathbb{E}_\rho(u_0)\big\}$.

In the case $\rho \notin \mathrm{L}^1(\mathbb{R}^N)$, we take a compact set K such that

$$\int_K \rho(x)\,\mathrm{d}x > \int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x\,,$$

which is always possible since $u_0 \in \mathrm{L}^1(\rho)$. Using Proposition 4.1-(i), Lemma 5.4 and Fatou's Lemma, we obtain

$$\begin{aligned} u_\infty \textstyle\int_K \rho(x)\,\mathrm{d}x &\leqslant \liminf_{n\to\infty} \textstyle\int_K \rho(x)u_n(x,s+t_j)\,\mathrm{d}x \\ &\leqslant \liminf_{n\to\infty} \textstyle\int_{\mathbb{R}^N} \rho(x)u_n(x,s+t_j)\,\mathrm{d}x \\ &\leqslant \textstyle\int_{\mathbb{R}^N} \rho(x)u_0(x)\,\mathrm{d}x\,. \end{aligned}$$

This implies that necessarily $u_\infty = 0$, hence the ω-limit set is reduced to $\{0\}$. □

5.5 Proofs of Theorems 1.2 and 1.3

As consequence of the fact that the $\omega-$limit is given by only one function we can pass to the limit in the time variable without extracting any subsequence. Then it only remains to check that the convergence is better, which is done by using transform $\mathcal{T}_\rho$.

Proof of Theorem 1.3. Under the hypotheses of the Theorem, we have that u_0 is continuous, bounded, and ρ not integrable, but nevertheless $u_0 \in \mathrm{L}^1(\rho)\cap\mathrm{L}^2(\rho)$. Thus Lemmas 5.1 and 5.5 imply that for any $s > 0$, at least along a subsequence $t_n \to \infty$ we have $u(x,s+t_n) \to 0$ in L^∞-weak*. But the same arguments are valid for any other subsequence such that $u(x,s+t'_n)$ converges weakly. Since the limit is always zero, we deduce that for any $s > 0$,

$$u(x,s+t) \xrightarrow[t\to\infty]{\mathrm{L}^\infty\text{-weak*}} 0\,,$$

which implies that $\big(J * u(s+t)\big)$ converges strongly in $\mathrm{L}^1_{\mathrm{loc}}$ as $t \to \infty$. Then,

$$\rho(x)\partial_s u(x,s+t) = \big(J * u(s+t)\big)(x) - u(x,s+t) \xrightarrow[t\to\infty]{\mathrm{L}^\infty\text{-weak*}} 0\,.$$

Even more, from lemma 5.2 we obtain that for any compact set K,

$$\int_t^\infty \left(\int_K \rho(x)|u_t|(x,s)\,\mathrm{d}x\right)^2 \mathrm{d}s \leqslant \int_t^\infty \left(\int_K \rho(x)|u_t|^2(x,s)\,\mathrm{d}x\right)\left(\int_K \rho(x)\,\mathrm{d}x\right)\mathrm{d}s \leqslant C(K,u_0,\rho).$$

Then, at least for some sequence $t_k \to +\infty$, we have $\rho(x)\partial_s u(x,s+t_k) \to 0$ in $\mathrm{L}^1_{\mathrm{loc}}$. Summing up, we obtain that

$$\lim_{t_k\to+\infty} u(x,s+t_k) = 0 \text{ in } \mathrm{L}^1_{\mathrm{loc}}\,.$$

Of course, if $t \mapsto u(x,s+t)$ were to converge in $\mathrm{L}^1_{\mathrm{loc}}$ along another subsequence $t'_k \to \infty$, the limit would necessarily be zero, so that finally $u(\cdot,t) \to 0$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^N)$ as $t \to \infty$. Moreover, since $t \mapsto u(\cdot,t)$ remains bounded in $\mathrm{L}^\infty(\mathbb{R}^N)$, we deduce that the convergence holds in $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N)$ for any $1 \leqslant p < \infty$. □

Proof of Theorem 1.2. The first part is done exactly as in the proof of Theorem 1.3, except that ρ is integrable here so that the limit is not zero, but $\mathbb{E}_\rho(u_0)$. To end the proof in this case, it only remains to prove the $\mathrm{L}^1(\rho)$ convergence. We fix $\varepsilon > 0$ and choose $R > 0$ big enough so that (remember that ρ is integrable):

$$\int_{|x|>R} \rho(x)\,\mathrm{d}x \leqslant \varepsilon\,.$$

Then

$$\int_{\mathbb{R}^N} |u(x,t) - u_\infty|\rho(x)\,\mathrm{d}x \leqslant 2\varepsilon\|u_0\|_\infty + \int_{|x|\leqslant R} |u(x,t) - u_\infty|\rho(x)\,\mathrm{d}x\,,$$

and using the $\mathrm{L}^1_{\mathrm{loc}}$ convergence we get:

$$\limsup_{t\to\infty} \int_{\mathbb{R}^N} |u(x,t) - u_\infty|\rho(x)\,\mathrm{d}x \leqslant 2\varepsilon\|u_0\|_\infty\,.$$

Since ε is arbitrary, we get that the limit is zero, which ends the proof. □

6 Unbounded solutions

In this section we derive some results for unbounded initial data and solutions. Let us mention that in the case $\rho \equiv 1$, further results are to be found in [1]. But here we still face the problem of the space inhomogenity implied by ρ.

6.1 An existence result for quadratic growth

We assume that there exists some $\eta > 0$ and $0 \leqslant \gamma \leqslant 2$ such that for any $x \in \mathbb{R}^N$,

$$\rho(x) \geqslant \frac{\eta}{1 + |x|^\gamma}\,. \tag{10}$$

This hypothesis means that ρ does not degenarate too rapidly at infinity. We then begin by constructing a supersolution of the problem (recall that $\mathbb{V}(J)$ is the second-order momentum of J, which is a finite constant):

Lemma 6.1 *If* (10) *holds, then for any $\lambda \geqslant \mathbb{V}(J)/\eta$ and $A > 0$ the following function is a (classical) supersolution of* (1)*:*

$$\mathcal{U}(x,t) := A\,\mathrm{e}^{\lambda t}(1 + |x|^2)\,. \tag{11}$$

Proof. It is a simple calculation: $\partial_t \mathcal{U} = \lambda A\,\mathrm{e}^{\lambda t}(1 + |x^2|)$,

$$\begin{aligned} J * \mathcal{U} - \mathcal{U} = &\; A\,\mathrm{e}^{\lambda t}(J * |x|^2 - |x|^2) \\ = &\; A\,\mathrm{e}^{\lambda t}\left(\int_{\mathbb{R}^N} J(y)|y|^2\,\mathrm{d}y - 2\int_{\mathbb{R}^N} J(y) < x, y > \mathrm{d}y\right) \\ = &\; A\,\mathrm{e}^{\lambda t}\mathbb{V}(J)\,. \end{aligned}$$

Hence we have a supersolution provided $\lambda\rho(x)(1 + |x|^\gamma) \geqslant \mathbb{V}(J)$. Since by assumption we have $\rho(x)(1 + |x|^\gamma) \geqslant \eta$, it is enough to impose $\lambda\eta \geqslant \mathbb{V}(J)$, hence the result. □

Proposition 6.1 *Let us assume that* (10) *holds. Then for any nonnegative* $u_0 \in \mathrm{C}^0(\mathbb{R}^N)$*, satisfying* $u_0(x) \leqslant C(1+|x|^2)$ *for some* $C > 0$*, there exist a strong solution* u *of* (1) *with* $u(x,0) = u_0(x)$*.*

Proof. Let us first consider an approximation $u_{0n} = u_0 \cdot \chi_n$ where χ_n is smooth, nonnegative, compactly supported and $\chi_n \nearrow 1$. Let u_n be the unique solution of (1) with initial data u_{0n} given by Proposition 3.1, then by applying the comparison result for bounded solutions, the sequence u_n is nondecreasing.

On the other hand, if A is big enough, and for instance $\lambda = \mathbb{V}(J)/\eta$, we may use the supersolution $\mathcal{U}$ defined in (11) to compare with u_n. Notice that $\mathcal{U}$ is not bounded, but this is allowed in Lemma 2.2, which gives:

$$u_n(x,t) \leqslant A\,\mathrm{e}^{\lambda t/\eta}(1+|x|^2)\,.$$

Hence the sequence u_n converges to some u and we are able to pass to the limit in $J * u_n$ by dominated convergence, using that $\mathcal{U}$ is integrable with respect to translations of J.

Using now Lemma 2.1, we deduce that u is a classical solution of (1) and the initial data of u is u_0. □

Remark 6.1 This construction does in fact give a minimal solution: if u_1 is any other solution, then it can be used as a supersolution for any u_n and passing to the limit shows that $u \leqslant u_1$. One can think that if we restrict the initial data to grow at most like $|x|^{2-\varepsilon}$, then uniqueness holds because the comparison argument is valid in this class (see Remark 2.2). However, it is not clear whether the constructed solution enters this class unless we know more about u_0, see [1].

6.2 Asymptotic behaviour for unbounded solutions when ρ is integrable

We prove now that if u_0 is integrable with respect to ρ, the isothermalisation phenomenon occurs (whether infinite or not). Notice that we gave sufficient conditions for existence of a minimal solution in the previous section. The first result is the following:

Proposition 6.2 *Let* $u_0 \in \mathrm{C}^0(\mathbb{R}^N) \cap \mathrm{L}^1(\rho)$*,* $\rho \in \mathrm{L}^1(\mathbb{R}^N)$ *and assume there exists a solution* u *such that* $u(x,0) = u_0(x)$*. Then, if* $\underline{u}$ *denotes the minimal solution, we have*

$$\lim_{t\to\infty} \int_{\mathbb{R}^N} |\underline{u}(x,t) - \mathbb{E}_\rho(u_0)|\,\rho(x)\,\mathrm{d}x = 0\,.$$

Proof. Let $\underline{u}$ be the minimal solution and let us use the same monotone approximations that were used in Proposition 6.1. Since u_n is bounded, Lemma 2.2 implies that we have a bound from above:

$$u(x,t) \geqslant u_n(x,t) \text{ in } \mathbb{R}^N \times [0,\infty)\,.$$

But since $u_n(x,0) \in \mathrm{L}^\infty(\mathbb{R}^N)$, Proposition 4.1 implies

$$\int_{\mathbb{R}^N} u_n(x,t)\rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} u_n(x,0)\rho(x)\,\mathrm{d}x.$$

Moreover, the convergence of u_n to $\underline{u}$ is monotone, so we can pass to the limit in the above equation to obtain

$$\int_{\mathbb{R}^N} \underline{u}(x,t)\rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \underline{u}(x,0)\rho(x)\,\mathrm{d}x.$$

Using the above three equations we get that

$$\int_{\mathbb{R}^N} |\underline{u}(x,t) - \mathbb{E}_\rho(u_0)|\rho(x)\,\mathrm{d}x \leqslant I_1 + I_2 + I_3,$$

where,

$$I_1 = \int_{\mathbb{R}^N} |\underline{u}(x,t) - u_n(x,t)|\rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} (\underline{u}(x,0) - u_n(x,0))\rho(x)\,\mathrm{d}x\,.$$

$$I_2 = \int_{\mathbb{R}^N} |u_n(x,t) - \mathbb{E}_\rho(u_0\chi_n)|\rho(x)\,\mathrm{d}x,$$

$$I_3 = \int_{\mathbb{R}^N} |\mathbb{E}_\rho(u_0\chi_n) - \mathbb{E}_\rho(u_0)|\rho(x)\,\mathrm{d}x.$$

Observe that I_1 and I_3 are independents of t and tend to zero as $n \to \infty$. Moreover, u_n satisfies the hypothesis of Theorem 1.2 so that I_2 tends to zero as $t \to \infty$. Therefore, we first have

$$\limsup_{t\to+\infty} \int_{\mathbb{R}^N} |\underline{u}(x,t) - \mathbb{E}_\rho(u_0)|\rho(x)\,\mathrm{d}x \leqslant I_1 + I_3\,,$$

so that taking the limit as $n \to \infty$ yields the desired result. □

In the case when $u_0 \notin \mathrm{L}^1(\rho)$, then infinite isothermalisation occurs:

Proposition 6.3 *Let $\rho \in \mathrm{L}^1(\mathbb{R}^N)$ and $u_0 \in \mathrm{C}^0(\mathbb{R}^N)$ such that $u_0 \notin \mathrm{L}^1(\rho)$. Then for any solution u with initial data u_0 and any $1 \leqslant p < \infty$, the following asymptotic behaviour holds:*

$$\lim_{t\to+\infty} u(x,t) = +\infty \text{ in } \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N)\,.$$

Proof. As before, if there exists a solution, then we can approximate the minimal solution $\underline{u}$ by the family u_n used in Proposition 6.1. Since this approximation is monotone,

$$\underline{u}(x,t) \geqslant u_n(x,t) \text{ in } \mathbb{R}^N \times [0,\infty)\,.$$

But $u_n(x,0)$ satisfies the hypotheses of Theorem 1.2 so that

$$\liminf_{t\to+\infty} \underline{u}(x,t) \geqslant \lim_{t\to+\infty} u_n(x,t) = c_n\,,$$

where $c_n = \mathbb{E}_\rho(u_n(x,0))$, the limit holding in all $\mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N)$. Hence passing to the limit as $n \to +\infty$, we obtain the result for $\underline{u}$ since $c_n \to \mathbb{E}_\rho(u_0) = +\infty$, thus the same holds for any other solution. □

Theorems 1.4 and 1.5 follow from the conjunction of Propositions and 6.1, 6.2 and 6.3.

ACKNOWLEDGEMENTS

The authors is partially supported by project MTM2008-06326-C02-02 (Spain). R. Ferreira is also partially supported by grant GR58/08-Grupo 920894.

References

[1] C. Brändle, E. Chasseigne, R. Ferreira *Unbounded solutions of the nonlocal heat equation*, preprint.

[2] E. Chasseigne *The Dirichlet problem for some nonlocal diffusion equations*, Differential Integral Equations **20** (2007), 1389–1404.

[3] E. Chasseigne, M. Chaves, J. Rossi *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl. **86** (2006), 271–291.

[4] E. Chasseigne, F. Quiros, *Deconcentration effects in Local and Non-local heat equations*, in preparation (2009).

[5] D. Eidus, *The Cauchy problem for the nonlinear filtration equation in an inhomogeneous medium*, J. Differential Equations **84** (1990), 309–318.

[6] D. Eidus, *The perturbed Laplace operator in a weighted L^2-space*, J. Funct. Anal. **100** (1991), 400–410.

[7] P. Fife, *Some nonclassical trends in parabolic and parabolic-like evolutions.* Trends in nonlinear analysis, 153–191, Springer, Berlin, 2003.

[8] S. Kamin and P. Rosenau, *Nonlinear diffusion in a finite mass medium*, Comm. Pure Appl. Math. **35** (1982), 113–127.

[9] S. Kamin, *Heat propagation in an inhomogeneous medium*, Progress in partial dierential equations: the Metz Surveys **4**, 229-237, Pitman Res. Notes Math. Ser. **345**, Longman, Harlow, 1996.

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2009

1. DESIGN OF CODE DIVISION MULTIPLE ACCESS FILTERS USING GLOBAL OPTIMIZATION TECHNIQUES, B. Ivorra, B. Mohammadi, and A. M.Ramos
2. DYNAMICS IN DUMBBELL DOMAINS II. THE LIMITING PROBLEM, J.M. Arrieta, A. N. Carvalho and G. Lozada-Cruz
3. DYNAMICS IN DUMBBELL DOMAINS III. CONTINUITY OF ATTRACTORS , J. M. Arrieta, A. N. Carvalho and G. Lozada-Cruz
4. GEOMETRIC VERSUS SPECTRAL CONVERGENCE FOR THE NEUMANN LAPLACIAN UNDER EXTERIOR PERTURBATIONS OF THE DOMAIN , J. M. Arrieta and D. Krejcirik
5. ON THE MODELLING AND SIMULATION OF HIGH PRESSURE PROCESSES AND INACTIVATION OF ENZYMES IN FOOD ENGINEERING, J.A. Infante, B. Ivorra, Á.M. Ramos and J.M. Rey
6. CHARACTERIZATION OF THE MARTIAN SURFACE LAYER, G.Martínez, F. Valero, L. Vazquez.
7. CHARACTERIZATION OF THE MARTIAN CONVECTIVE BOUNDARY LAYER, G.Martínez, F. Valero, L. Vazquez.
8. INFINITE RESONANT SOLUTIONS AND TURNING POINTS IN A PROBLEM WITH UNBOUNDED BIFURCATION, J.M. Arrieta, R.Pardo, A.Rodríguez—Bernal
9. CASCADES OF HOPF BIFURCATIONS FROM BOUNDARY DELAY, J.M. Arrieta, N. Cónsul, S. Oliva
10. QUENCHING PHENOMENA FOR A NON-LOCAL DIFFUSION EQUATION WITH A SINGULAR ABSORPTION, R. Ferreira
11. WELL-POSEDNESS OF THE EINSTEIN–EULER SYSTEM IN ASYMPTOTICALLY FLAT SPACETIMES: THE CONSTRAINT EQUATIONS, U. Brauer and L. Karp
12. WELL-POSEDNESS OF THE EINSTEIN–EULER SYSTEM IN ASYMPTOTICALLY FLAT SPACETIMES: THE EVOLUTION EQUATIONS, U. Brauer and L. Karp
13. EXTREMAL EQUILIBRIA FOR MONOTONE SEMIGROUPS IN ORDERED SPACES WITH APPLICATIONS TO EVOLUTIONARY EQUATIONS, J. W. Cholewa and A. Rodriguez-Bernal
14. ASYMPTOTIC BEHAVIOUR OF A PARABOLIC PROBLEM WITH TERMS CONCENTRATED IN THE BOUNDARY, A. Jiménez-Casas and A. Rodriguez-Bernal
15. ASYMPTOTIC BEHAVIOR OF REACTION DIFFUSION EQUATIONS IN WEIGHTED SOBOLEV SPACES, J.M. Arrieta, N. Moya and A. Rodriguez-Bernal

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2010

1. CONTINUITY OF DYNAMICAL STRUCTURES FOR NON-AUTONOMOUS EVOLUTION EQUATIONS UNDER SINGULAR PERTURBATIONS, J. Arrieta, A. N. Carvalho, J. Langa and A. Rodríguez-Bernal.

2. ON THE LONG TIME BEHAVIOUR OF NON-AUTONOMOUS LOTKA-VOLTERRA MODELS WITH DIFFUSION VIA THE SUB-SUPER TRA JECTORY METHOD, J.A. Langa, A. Rodríguez-Bernal and A. Suárez.

3. MODELLING AND SIMULATION OF A POLLUTED WATER PUMPING PROCESS, C. Alavani, R. Glowinski, S. Gomez, B.Ivorra, P. Joshi and A. M. Ramos.

4. EXPANDING THE ASYMPTOTIC EXPLOSIVE BOUNDARY BEHAVIOR OF LARGE SOLUTIONS TO A SEMILINEAR ELLIPTIC EQUATION, S. Alarcón, G. Díaz, R. Letelier and J. M. Rey.

5. A SINGULAR PERTURBATION IN A LINEAR PARABOLIC EQUATION WITH TERMS CONCENTRATING ON THE BOUNDARY, A. Rodríguez-Bernal.

6. ISOTHERMALISATION FOR A NON-LOCAL HEAT EQUATION, E. Chasseigne and R. Ferreira.