PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2011-2

Infinitely many stability switches in a problem with sublinear oscillatory boundary conditions

A. Castro and R. Pardo

Febrero-2011

http://www.mat.ucm.es/deptos/ma
e-mail:matemática_aplicada@mat.ucm.es

INFINITELY MANY STABILITY SWITCHES IN A PROBLEM WITH SUBLINEAR OSCILLATORY BOUNDARY CONDITIONS

ALFONSO CASTRO, ROSA PARDO

Abstract. We consider an elliptic equation $-\Delta u + u = 0$ with nonlinear boundary conditions $\frac{\partial u}{\partial n} = \lambda u + g(\lambda, x, u)$, where $\frac{g(\lambda,x,s)}{s} \to 0,$ as $|s| \to \infty$ and g is oscillatory. We provide sufficient conditions on g for the existence of unbounded sequences of stable solutions, unstable solutions, and turning points.

1. Introduction

In this paper we consider solutions to the elliptic problem with nonlinear boundary conditions

$$\begin{cases} -\Delta u + u &= 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} &= \lambda u + g(\lambda, x, u), & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

in a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$. Throughout this paper we assume:

(H1): $g : \mathbb{R} \times \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function (i.e. $g = g(\lambda, x, s)$ is measurable in $x \in \Omega$, and continuous with respect to $(\lambda, s) \in \mathbb{R} \times \mathbb{R}$). Moreover, there exist $G_1 \in L^r(\partial\Omega)$ with $r > N - 1$ and continuous functions $\Lambda : \mathbb{R} \to \mathbb{R}^+$, and $U : \mathbb{R} \to \mathbb{R}^+$, satisfying

$$\begin{cases} \|g(\lambda, x, s)| \leq \Lambda(\lambda) G_1(x) U(s), & \forall (\lambda, x, s) \in \mathbb{R} \times \partial\Omega \times \mathbb{R}, \\ \limsup_{|s|\to\infty} \dfrac{U(s)}{|s|^\alpha} < +\infty & \text{for some} \quad \alpha < 1. \end{cases}$$

(H2) **:** The partial derivative $g_s(\lambda, \cdot, \cdot) \in C(\partial\Omega \times \mathbb{R})$ where $g_s := \frac{\partial g}{\partial s}$, and there exist $F_1 \in L^r(\partial\Omega)$, with $r > N - 1$, and $\rho < 1$ such that

$$\frac{|g(\lambda, x, s) - s g_s(\lambda, x, s)|}{|s|^\rho} \leq F_1(x), \qquad \text{as } \lambda \to \sigma_1 \tag{1.2}$$

for $x \in \partial\Omega$ and $s \gg 1$ sufficiently large.

(H3) **:** The second partial derivative $g_{ss}(\lambda, \cdot, \cdot) \in C(\partial\Omega \times \mathbb{R})$ is such that

$$\sup_{|s|\geq M} \left\| \frac{g_{ss}(\lambda, \cdot, s)}{|s|^{\rho-\alpha-1}} \right\|_{L^\infty(\partial\Omega)} \to 0 \qquad \text{as } M \to \infty \quad \text{and } \lambda \to \sigma_1. \tag{1.3}$$

2000 *Mathematics Subject Classification.* 35B32, 35B34, 35B35, 58J55, 35J25, 35J60, 35J65.

Key words and phrases. Resonance, stability, instability, multiplicity, bifurcation from infinity, sublinear oscillating boundary conditions, turning points.

The second author is supported Spanish Ministerio de Ciencia e Innovacion (MICINN) under Project MTM2009-07540, by UCM-BSCH, Spain, GR58/08, Grupo 920894 and also by Programa Becas Complutense del Amo. This work was elaborated during a sabbatical visit of the second author to the Department of Mathematics, Harvey Mudd College, Claremont, USA, whose hospitality she thanks.

Let $\{\sigma_i\}_{i=1}^{\infty}$ denote the sequence of *Steklov* eigenvalues of the problem

$$\left\{\begin{array}{rcll} -\Delta\Phi+\Phi &=& 0, & \text{in } \Omega \\ \frac{\partial\Phi}{\partial n} &=& \sigma\Phi, & \text{on } \partial\Omega. \end{array}\right. \tag{1.4}$$

The Steklov eigenvalues form an increasing sequence of real numbers, $\{\sigma_i\}_{i=1}^{\infty}$. Each eigenvalue has finite multiplicity. The first eigenvalue σ_1 is simple and, due to Hopf's Lemma, we may assume its eigenfunction Φ_1 to be strictly positive in $\overline{\Omega}$. The eigenfunctions are orthogonal in $L^2(\partial\Omega)$ and we take $\|\Phi_1\|_{L^\infty(\partial\Omega)}=1$.

As stated in [1, Theorem 3.4], due to (H1) there exists a connected set of positive solutions of (1.1). We denote it by $\mathcal{D}^+\subset\mathbb{R}\times C(\bar\Omega)$, and recall that for $(\lambda,u_\lambda)\in\mathcal{D}^+$

$$u=s\Phi_1+w,\qquad \text{with } \ w=o\left(|s|\right) \quad \text{and} \quad |\sigma_1-\lambda|=o(1) \quad \text{as } |s|\to\infty.$$

The set $\mathcal{D}^+$ is known as a *branch bifurcating from infinity* in the sense of Rabinowitz, see [10, 1].

For $(\lambda,u_\lambda)\in\mathcal{D}^+$ we say that u_λ is a *stable* solution if there exists a neighborhood of u_λ in $C(\bar\Omega)$ such that for initial data in that neighborhood the solution to the parabolic problem

$$\left\{\begin{array}{rcll} u_t-\Delta u+u &=& 0, & \text{in } \Omega\times\mathbb{R}^+ \\ \frac{\partial u}{\partial n} &=& \lambda u+g(\lambda,x,u), & \text{on } \partial\Omega\times\mathbb{R}^+, \\ u(0,x) &=& u_0(x), & \text{in } \Omega. \end{array}\right. \tag{1.5}$$

converges to u_λ as $t\to+\infty$. On the other hand we say that u_λ is *unstable* if any neighborhood of u_λ contains initial conditions such the solution to (1.5) leaves that neighborhood in finite time. That is asymptotic stability in the Lyapunov sense.

Definition 1.1. *A solution (λ^*,u^*) of (1.1) in the branch of solutions $\mathcal{D}^+\subset\mathbb{R}\times C(\bar\Omega)$ is called a* **turning point** *if there is a neighborhood W of (λ^*,u^*) in $\mathbb{R}\times C(\bar\Omega)$ such that, either $W\cap\mathcal{D}^+\subset[\lambda^*,\infty)\times C(\bar\Omega)$ or $W\cap\mathcal{D}^+\subset(-\infty,\lambda^*]\times C(\bar\Omega)$.*

Our goal is to give conditions on the sublinear oscillatory term g that guarantee the existence of unbounded sequences of stable solutions, unstable solutions and turning points.

Our main result, Theorem 1.3 below, is exemplified by the case in which

$$g(x,s):=s^\alpha\left[\sin\left(\left|\frac{s}{\Phi_1(x)}\right|^\beta\right)+C\right]\qquad \text{with } \alpha<1, \tag{1.6}$$

In fact we have:

Theorem 1.2. *Assume that g is given by (1.6). If*

$$\beta>0 \qquad \textit{and} \quad \alpha+\beta<1,$$

then the unbounded branch of positive solutions of (1.1) contains a sequence of stable solutions, a sequence of unstable solutions and a sequence of turning points.

The proof of this Theorem follows directly from Theorem 1.3.

In Figures 1 and 2 we plot the bifurcation diagram in the one dimensional case for g as above. Figure 3 sketches the changes of stability of solutions.

Our main result is the following.

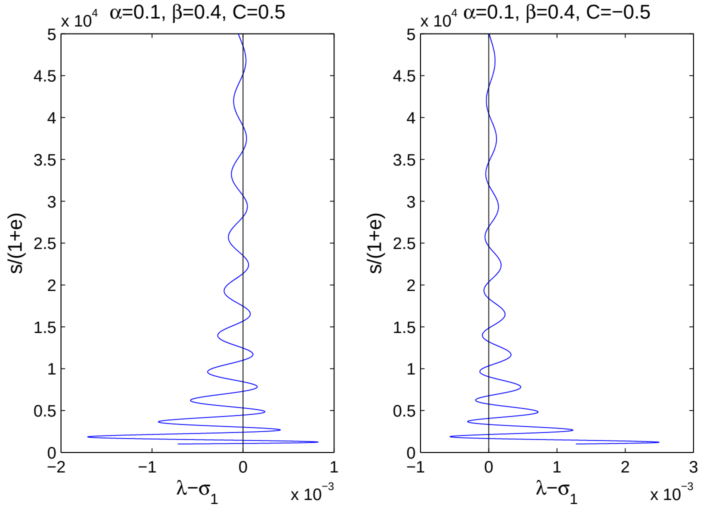

FIGURE 1. Bifurcation diagram having infinitely many sub-critical solutions ($\lambda < \sigma_1$), super-critical solutions ($\lambda > \sigma_1$), stable solutions, unstable solutions, turning points and resonant solutions ($\lambda = \sigma_1$).

Theorem 1.3. *Assume the nonlinearity g satisfies hypothesis* (H1), (H2) *and* (H3).
Assume also that

$$\sup_{|s|\geq M}\left|\frac{g(\lambda,x,s)-sg_s(\lambda,x,s)}{|s|^\rho}-\frac{g(\sigma_1,x,s)-sg_s(\sigma_1,x,s)}{|s|^\rho}\right|\to 0 \quad as \quad s\to\infty, \tag{1.7}$$

pointwise in x, for $M \gg 1$.
Let $F : \mathbb{R}\times C(\bar\Omega)\to\mathbb{R}$ be defined by

$$F(\lambda,u):=\int_{\partial\Omega}\frac{ug(\lambda,\cdot,u)-u^2g_s(\lambda,\cdot,u)}{|u|^{1+\rho}}\Phi_1^{1+\rho}. \tag{1.8}$$

If there exist sequences $\{s_n\}$, $\{s'_n\}$ converging to $+\infty$, such that

$$\lim_{n\to+\infty}F(\sigma_1,s'_n\Phi_1) \ <0< \lim_{n\to+\infty}F(\sigma_1,s_n\Phi_1), \tag{1.9}$$

then

(i) There exists a sequence $\{(\lambda_n,u_n)\}\in\mathcal{D}^+$ of stable solutions to (1.1) *and a sequence $\{(\lambda'_n,u'_n)\}\in\mathcal{D}^+$ of unstable solutions such that $(\lambda_n,\|u_n\|_{L^\infty(\partial\Omega)})\to(\sigma_1,\infty)$ and $(\lambda'_n,\|u'_n\|_{L^\infty(\partial\Omega)})\to(\sigma_1,\infty)$ as $n\to\infty$.*
(ii) There exists a sequence $\{(\lambda^*_n,u^*_n)\}\in\mathcal{D}^+$ of turning points such that $(\lambda^*_n,\|u^*_n\|_{L^\infty(\partial\Omega)})\to(\sigma_1,\infty)$as $n\to\infty$.

Our result is sharp in that if condition (1.9) fails, all solutions in $\mathcal{D}^+$ may be either stable or unstable for s big enough, see [2, Theorem 3.4]. Our result proves the existence of infinitely many turning points, even in the absence of resonant solutions, see Figure 2. There it can be seen that the unbounded sequence of turning points given by Theorem 1.3 can be either *subcritical* (i.e. for values of the parameter $\lambda < \sigma_1$), , see Figure 2 left, or *supercritical* (i.e. for $\lambda > \sigma_1$), see Figure 2 right, or may have a sequence of subcritical solutions as well a sequence of supercritical solutions. Hence, by connectedness of $\mathcal{D}^+$, the branch contains infinitely many *resonant* solutions (i.e. for $\lambda = \sigma_1$), see Figure 1.

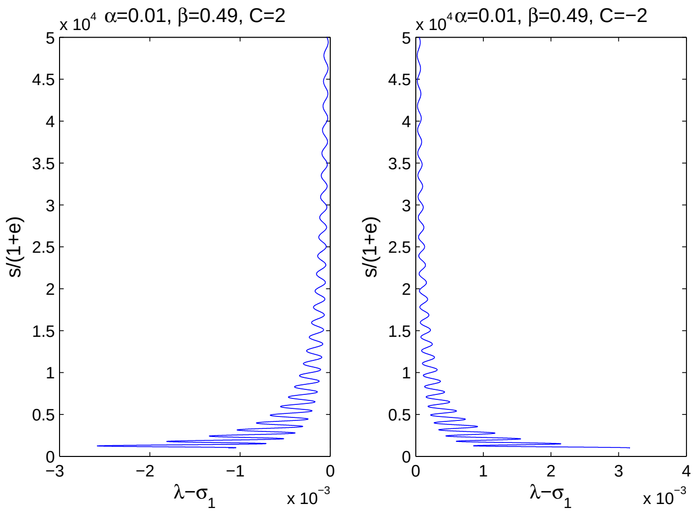

FIGURE 2. A bifurcation diagram of stable and unstable solutions, on the left all of them are subcritical, on the right all of them are supercritical, and none is resonant.

Related results for the case of a nonlinear reaction in Ω and homogeneous Dirichlet boundary conditions were established in [4, 5, 6, 9]. In [6] the authors work in the unit ball $B \subset \mathbb{R}^N$ with $N \geq 1$ and the nonlinear term is $\lambda u + \sin(u)$. They proved that when $\lambda = \lambda_1$, the first eigenvalue with Dirichlet boundary conditions, the problem has infinitely many solutions for $1 \leq N \leq 5$ and at most finitely many solutions for $N \geq 6$. We refer the reader to [7, 8] for problems related with nonlinear boundary conditions.

This paper is organized as follows. In Section 2 we collect some essentially known results on Lyapunov stability. Section 3 contains the proof of our main result, giving sufficient conditions for having stable and unstable solutions. Finally Section 4 presents two examples, the typical oscillatory nonlinearity (1.6) and the one dimensional case.

2. Lyapunov function and stability

For λ fixed we consider

$$I(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) - \frac{\lambda}{2}\int_{\partial\Omega} u^2 - \int_{\partial\Omega} G(\lambda, \cdot, u)$$

where $G(\lambda, x, s) := \int_{s_0}^{s} g(\lambda, x, t)\,dt$ for some $s_0 \gg 1$ fixed. An elementary calculation shows that if u is a solution to the parabolic equation (1.5) then $\frac{d}{dt} I(u(t)) = I'(u(t))u_t \leq 0$, i.e., I is Lyapunov function for the parabolic problem (1.5).

Moreover, if u_λ is a solution to (1.1), then it is a critical point for I. Furthermore, u_λ is stable if the quadratic form

$$Q_{u_\lambda}(v, w) := \int_\Omega \nabla v \cdot \nabla w + vw - \int_{\partial\Omega} \lambda vw + g_s(\lambda, \cdot, u_\lambda)vw.$$

is positive definite. On the other hand if Q_{u_λ} is negative definite in one direction then u_λ is unstable. Thus we have

Lemma 2.1. *If $\mu_1 \equiv \mu_1(\lambda, u_\lambda)$ denotes the principal eigenvalue of*

$$\left\{ \begin{array}{rcll} -\Delta\varphi_1 + \varphi_1 & = & 0, & \text{in } \Omega \\ \frac{\partial\varphi_1}{\partial n} & = & \mu_1\varphi_1 + g_s(\lambda, x, u_\lambda)\varphi_1, & \text{on } \partial\Omega \end{array} \right. \tag{2.1}$$

then u_λ is stable, if $\mu_1 > \lambda$. Also u_λ is unstable if $\mu_1 < \lambda$.

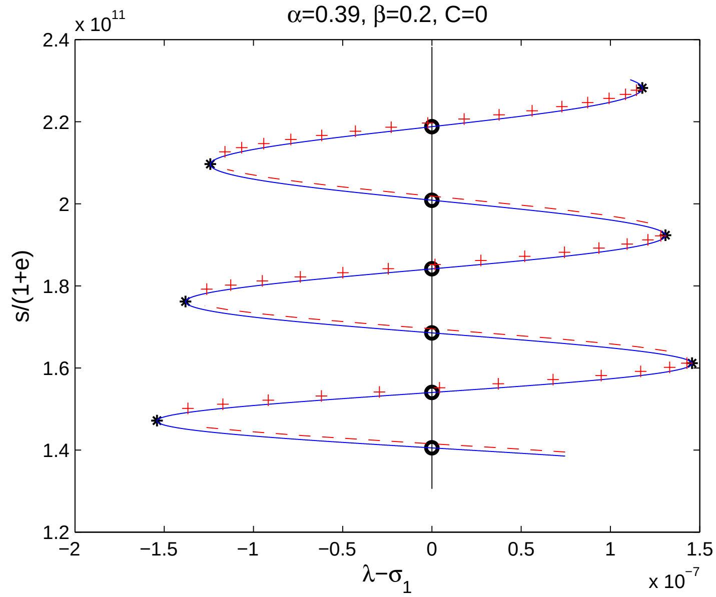

FIGURE 3. Bifurcation diagram and sketch of the stability of solutions, + for stable solutions and − for unstable solutions. There are marks $*$ for turning points and o for resonant solutions.

Proof. Suppose $\mu_1 > \lambda$. The variational characterization of μ_1 states that

$$\mu_1 := \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 + u^2 - \int_{\partial\Omega} g_s(\lambda, \cdot, u_\lambda) u^2}{\int_{\partial\Omega} u^2}$$

Therefore, for any $u \in H^1(\Omega) - \{0\}$, we have

$$\begin{aligned} 0 &\leq \int_\Omega |\nabla u|^2 + u^2 - \int_{\partial\Omega} \mu_1 u^2 + g_s(\lambda, \cdot, u_\lambda) u^2 \\ &< \int_\Omega |\nabla u|^2 + u^2 - \int_{\partial\Omega} \lambda u^2 + g_s(\lambda, \cdot, u_\lambda) u^2. \end{aligned}$$

Hence Q_{u_λ} is positive definite and u_λ is stable.

On the other hand, if $\mu_1 < \lambda$, letting φ_1 denote the eigenfunction corresponding to the eigenvalue μ_1, then

$$\begin{aligned} 0 &= \int_\Omega \|\nabla \varphi_1\|^2 + \varphi_1^2 - \int_{\partial\Omega} \mu_1 \varphi_1^2 + g_s(\lambda, \cdot, u_\lambda) \varphi_1^2 \\ &> \int_\Omega \|\nabla \varphi_1\|^2 + \varphi_1^2 - \int_{\partial\Omega} \lambda \varphi_1^2 + g_s(\lambda, \cdot, u_\lambda) \varphi_1^2. \end{aligned}$$

Thus Q_{u_λ} is negative definite in the direction of φ_1, which proves that u_λ is unstable. □

3. AUXILIARY LEMMAS AND PROOF OF OUR MAIN RESULT

This section is devoted to giving sufficient conditions for the existence of unbounded sequences of stable solutions, unstable solutions, and turning points of (1.1).

Let α be the rate with which g goes to infinity, see (H1), and ρ be rate with which $g - s g_s$ goes to infinity, see (H2). In the first place we note that even if $\alpha \neq \rho$, then the *boundary Steklov eigenvalue* $\mu_1 \to \sigma_1$ and of the *boundary Steklov eigenfunction* $\varphi_1 \to \Phi_1$ as $\lambda \to \sigma_1$ and $\|u\|_{L^\infty(\partial\Omega)} \to \infty$. In the following Lemma we rewrite Lemma 3.2 in [2], the proof is exactly the same. The only restriction is that $\rho < 1$. We have the following result.

Lemma 3.1. *Assume the nonlinearity g satisfies hypotheses* (H1) *and* (H2).

Then for any sequence of solutions of (1.1), (λ_n, u_n) *such that* $\lambda_n \to \sigma_1$ *and* $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$, *setting* $\mu_{1,n} = \mu_1(\lambda_n, u_n)$, $\varphi_{1,n} = \varphi_1(\lambda_n, u_n)$, *the first eigenvalue and eigenfunction in* (2.1) *satisfy*

$$\mu_{1,n} \to \sigma_1 \qquad as \quad \lambda_n \to \sigma_1 \quad and \quad \|u_n\|_{L^\infty(\partial\Omega)} \to \infty, \tag{3.1}$$

$$\varphi_{1,n} \to \Phi_1 \quad in\ H^1(\Omega) \cap C^\beta(\overline{\Omega}) \quad as \quad \lambda_n \to \sigma_1 \quad and \quad \|u_n\|_{L^\infty(\partial\Omega)} \to \infty,$$

for some $\beta \in (0, 1)$.

Observe that (H1) and (H2) imply that,

$$\frac{|g_s(\lambda, x, s)|}{|s|^{\gamma-1}} \le |s|^{\rho-\gamma} F_1(x) + |s|^{\alpha-\gamma} G_1(x), \qquad \text{as } \lambda \to \sigma_1, \qquad \text{for } s \gg 1$$

where $\gamma = \max\{\rho, \alpha\} < 1$. Hence $\frac{|g_s(\lambda,x,s)|}{|s|^{\gamma-1}} \le D_1(x)$ with $D_1 \in L^r(\partial\Omega)$ with $r > N - 1$, for s big enough, $x \in \partial\Omega$ and $\lambda \to \sigma_1$.

In the second step, we analyze the changes of stability. To do that, we look at a detailed account of the asymptotic behavior of the nonlinear term

$$\underline{\mathbf{F}}_+ := \int_{\partial\Omega} \liminf_{(\lambda,s)\to(\sigma_1,+\infty)} \frac{sg(\lambda, \cdot, s) - s^2 g_s(\lambda, \cdot, s)}{|s|^{1+\rho}} \Phi_1^{1+\rho}$$

for $\rho < 1$. Changing lim inf by lim sup we define the number $\overline{\mathbf{F}}_+$. Assume $\alpha = \rho$, if

$$\underline{\mathbf{F}}_+ > 0, \qquad \text{then } \mathcal{D}^+ \text{ is stable and subcritical,}$$

see [2, Theorem 3.4], and if

$$\overline{\mathbf{F}}_+ < 0, \qquad \text{then } \mathcal{D}^+ \text{ is unstable and supercritical,}$$

see [2, Theorem 3.5]. In this paper we consider nonlinearities for which

$$\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+,$$

Unlike the case $\alpha = \rho$, , our assumption $\alpha \neq \rho$ allows for the existence of sequences of stable supercritical solutions and unstable subcritical solutions, see Theorem 1.3.

We shall argue as in [3] for the sub-critical and supercritical case. To determine whether a sequence of solutions (λ_n, u_n) is stable or unstable, one must check the sign of

$$\liminf_{n\to\infty} F(\lambda_n, u_n) \qquad \text{and of} \quad \limsup_{n\to\infty} F(\lambda_n, u_n). \tag{3.2}$$

where F is defined by (1.8). This is done in Lemma 3.2. But this requires an a priori knowledge of the solutions themselves, which is in general impracticable.

In [3, Proposition 3.2], it is proved that when g is such that

$$|g(\lambda, x, s)| = O\left(|s|^\alpha\right) \text{ as } |s| \to \infty \text{ for some } \alpha < 1,$$

then, the solutions in $\mathcal{D}^\pm$, can be described as

$$u_n = s_n\Phi_1 + w_n, \qquad \text{where} \quad \int_{\partial\Omega} w_n\Phi_1 = 0 \quad \text{and} \qquad w_n = O(|s_n|^\alpha) \text{ as } n \to \infty,$$

and we intend to unveil the signs in (3.2) by just looking at the signs of those lim inf at $\lambda_n = \sigma_1$ and $u_n = s_n\Phi_1$. This is achieved in Lemma 3.3.

With these tools, in Theorem 1.3 we take two sequences $\{s_n\}$ and $\{s'_n\}$ satisfying

$$-\infty < \lim_{n\to+\infty} F(\sigma_1, s'_n\Phi_1) < 0 < \lim_{n\to+\infty} F(\sigma_1, s_n\Phi_1) < \infty, \tag{3.3}$$

and from here we obtain the existence of unbounded sequences of stable and unstable solutions of (1.1) in $\mathcal{D}^+$.

Let us follow with the second technical lemma. Note that this result allows us to compare λ and μ_1 as $\lambda \to \sigma_1$. Next Lemma is essentially Lemma 3.3 in [2] rewritten for a different rate, we omit the proof.

Lemma 3.2. *Assume the nonlinearity g satisfies hypotheses* (H1) *and* (H2).

Then for any sequence of solutions of (1.1) (λ_n, u_n) *such that* $\lambda_n \to \sigma_1$ *and* $\|u_n\|_{L^\infty(\partial\Omega)} \to \infty$ *denoting by* $\mu_{1,n} = \mu_1(\lambda_n, u_n)$*, the first eigenvalue in* (2.1)*, we have, if* $u_n > 0$

$$
\begin{aligned}
\frac{\underline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2} &\leq \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \liminf_{n\to\infty} F(\lambda_n, u_n) \\
&\leq \liminf_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}} \leq \limsup_{n\to\infty} \frac{\mu_{1,n} - \lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}} \\
&\leq \frac{1}{\int_{\partial\Omega} {\Phi_1}^2} \limsup_{n\to\infty} F(\lambda_n, u_n) \leq \frac{\overline{\mathbf{F}}_+}{\int_{\partial\Omega} {\Phi_1}^2}
\end{aligned}
$$

A similar statement is obtained for the case $u_n < 0$, just changing $\underline{\mathbf{F}}_+$ by $\underline{\mathbf{F}}_-$ and $\overline{\mathbf{F}}_+$ by $\overline{\mathbf{F}}_-$.

In order to prove the main result, we have to guarantee that the signs in (3.2) can be deduced from those of (3.3). This is stated in the following technical result, which is a slight variation of [3, Lemma 3.3]

Lemma 3.3. *Assume that g satisfies hypotheses* (H1), (H2), (H3) *and* (1.7).

If $\lambda_n \to \sigma_1$, $s_n \uparrow \infty$ and there exists a constant C such that $\|w_n\|_{L^\infty(\partial\Omega)} \leq C|s_n|^\alpha$ for all $n \to \infty$, then

$$\liminf_{n\to+\infty} F(\lambda_n, s_n\Phi_1 + w_n) \geq \liminf_{n\to+\infty} F(\sigma_1, s_n\Phi_1),$$

where F is given by (1.8)*. Similarly*

$$\limsup_{n\to+\infty} F(\lambda_n, s_n\Phi_1 + w_n) \leq \limsup_{n\to+\infty} F(\sigma_1, s_n\Phi_1).$$

Proof. For short, let us denote by $h = g - sg_s$. For all $(\lambda, s) \approx (\sigma_1, +\infty)$ and for any $w \in L^\infty(\partial\Omega)$ such that $\frac{1}{2}\Phi_1 > \frac{|w|}{s}$, we have (with a constant C that may change from line to line)

$$
\begin{aligned}
\int_{\partial\Omega} |h(\lambda, \cdot, s\Phi_1 + w) - h(\lambda, \cdot, s\Phi_1)|\Phi_1 &\leq C\|w\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} \left| \int_0^1 h_s(\lambda, \cdot, s\Phi_1 + \tau w)\, d\tau \right| \\
&\leq C\|w\|_{L^\infty(\partial\Omega)} \sup_{\tau\in[0,1]} \|h_s(\lambda, \cdot, s\Phi_1 + \tau w)\|_{L^\infty(\partial\Omega)}
\end{aligned}
$$

Taking into account hypothesis (H3) and whenever $\|w\|_{L^\infty(\partial\Omega)} = O(|s|^\alpha)$, we deduce that

$$\int_{\partial\Omega} \frac{|h(\lambda, \cdot, s\Phi_1 + w) - h(\lambda, \cdot, s\Phi_1)|}{|s|^\rho} \Phi_1 \leq C \sup_{|s|\geq M} \left\| \frac{h_s(\lambda, \cdot, s)}{|s|^{\rho-\alpha}} \right\|_{L^\infty(\partial\Omega)} \longrightarrow 0 \tag{3.4}$$

as $\lambda \to \sigma_1$, $M \to +\infty$.

Consequently, for $\|w_n\|_{L^\infty(\partial\Omega)} = O(|s_n|^\alpha)$

$$
\begin{aligned}
&\liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\lambda_n, \cdot, s_n\Phi_1 + w_n)}{|s_n|^{1+\rho}} \Phi_1 \\
&\geq \lim_{\substack{\lambda\to\sigma_1\\ s\to+\infty}} \int_{\partial\Omega} \frac{sh(\lambda, \cdot, s\Phi_1 + w) - sh(\lambda, \cdot, s\Phi_1)}{|s|^{1+\rho}} \Phi_1 + \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\lambda_n, \cdot, s_n\Phi_1)}{|s_n|^{1+\rho}} \Phi_1 \\
&= \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\lambda_n, \cdot, s_n\Phi_1)}{|s_n|^{1+\rho}} \Phi_1 = \liminf_{n\to+\infty} \int_{\partial\Omega} \frac{s_n h(\sigma_1, \cdot, s_n\Phi_1)}{|s_n|^{1+\rho}} \Phi_1,
\end{aligned}
$$

where we used firstly (3.4) and secondly hypothesis (1.7).

Now note that the left hand side above can be written as

$$\frac{s_n h(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n|^{1+\rho}}\ \Phi_1=\frac{(s_n\Phi_1+w_n)h(\lambda_n,\cdot,s_n\Phi_1+w_n)}{|s_n\Phi_1+w_n|^{1+\rho}}\left|\Phi_1+\frac{w_n}{s_n}\right|^{\rho}\Phi_1.$$

Then, (H2) and the fact that $\Phi_1+w_n/s_n\to\Phi_1$ in $L^\infty(\partial\Omega)$ concludes the proof. □

We are now in a position to prove our main result, which states the existence of unbounded sequences of stable solutions, unbounded sequences of unstable solutions and also unbounded sequences of turning points.

Proof of Theorem 1.3

(i) To prove the result, we show that from (1.9) we can find two unbounded sequences of solutions $\{(\lambda_n,u_n)\}$, $\{(\lambda_n',u_n')\}$, with λ_n,λ_n' close enough to σ_1, such that $\mu_{1,n}:=\mu_1(\lambda_n,u_n)>\lambda_n$ and $\mu_{1,n}':=\mu_1(\lambda_n',u_n')<\lambda_n'$, respectively and then we use Lemma 2.1. We below focus in the stable case and the unstable one is analogous.

Since the projection of the unbounded branch of positive solutions on span$[\Phi_1]$, is an interval $[s_0,\infty)$, choose $(\lambda_n,u_n)\to(\sigma_1,\infty)$ such that

$$P(u_n):=\frac{\int_{\partial\Omega}u_n\Phi_1}{\int_{\partial\Omega}\Phi_1^2}=s_n,$$

with s_n as in (1.9). Writing $u_n=s_n\Phi_1+w_n$, from [3, Proposition 3.2] and hypotheses (H2), we obtain that $w_n=O(|s_n|^\alpha)$.

Taking into account Lemma 3.2 we have

$$\liminf_{n\to\infty}\frac{\mu_{1,n}-\lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}}\geq\liminf_{n\to\infty}\frac{\mu_{1,n}-\lambda_n}{\|u_n\|^{\rho-1}_{L^\infty(\partial\Omega)}}\geq\frac{1}{\int_{\partial\Omega}\Phi_1^2}\ \liminf_{n\to\infty}F(\lambda_n,u_n)\tag{3.5}$$

Applying Lemma 3.3 applied to the function $h=g-sg_s$, using firstly hypothesis (H1), (H2) and (1.7), and secondly (1.7) and (1.9), implies that

$$\liminf_{n\to\infty}F(\lambda_n,s_n\Phi_1+w_n)\geq\liminf_{n\to+\infty}F(\sigma_1,s_n\Phi_1)>0\tag{3.6}$$

The inequalities (3.5)-(3.6) imply that $\mu_{1,n}>\lambda_n$ for λ_n close enough to σ_1. Likewise it can be proved that $\mu_{1,n}'<\lambda_n'$ for λ_n' close enough to σ_1, ending this part of the proof.

(ii) To achieve this part of the proof, we use Leray-Schauder degree theory. Let

$$K_n:=\{(\lambda,u)\in\mathcal{D}^+ : P(u)=s\ \text{ and }\ s_n\leq s\leq s_n'\}.$$

For each $n\in\mathbb{N}$, K_n is a compact set in $\mathbb{R}\times C(\bar\Omega)$, see for instance [3, Proof of Theorem 3.4]. For each $n\in\mathbb{N}$ fix, let $\lambda_{min}:=\min\{\lambda:(\lambda,u)\in K_n\}$, and likewise λ_{max}. Assume to the contrary that K_n contains no turning point. In other words, assume that for each $\lambda\in[\lambda_{min},\lambda_{max}]$ there exist a unique solution $u_\lambda\in K_n$.

For any $b\in L^q(\partial\Omega)$, $q\geq1$, there exists a unique solution of

$$\begin{cases}-\Delta v+v &= 0, & \text{in }\Omega\\ \frac{\partial v}{\partial n} &= b, & \text{on }\partial\Omega.\end{cases}$$

Moreover $\|v\|_{W^{1,p}(\Omega)}\leq C\|b\|_{L^q(\partial\Omega)}$, with $p=q\frac{N}{N-1}$. We denote it by $T(b)=v$ and

$$S(b):=\gamma T(b),\qquad\text{where }\gamma:W^{1,p}(\Omega)\to W^{1-1/p,p}(\partial\Omega)\text{ is the trace operator.}$$

The operator S is known as the *Neumann-to-Dirichlet operator*. If $q>N-1$, then the map S transforms $L^q(\partial\Omega)$ into $C^\tau(\partial\Omega)$ for some $\tau\in(0,1)$, and is continuous and compact, see for instance [1, Lemma 2.1].

Let $H:[\lambda_{min},\lambda_{max}]\times C(\partial\Omega)\to C(\partial\Omega)$ be the homotopy defined by

$$H(\lambda,u):=\lambda Su+S(g(\lambda,\cdot,u)).$$

Hence, the fixed points of $H(\lambda,\cdot)$ are the solutions to (1.1). Let $\varepsilon>0$, writing $u=s\Phi_1+w$ and due to $\|w\|_{L^\infty(\partial\Omega)}=O(|s|^\alpha)$ with $\alpha<1$, we obtain $\|u-s\Phi_1\|_{L^\infty(\partial\Omega)}\le\varepsilon s$ for any s big enough.

Now consider the Leray-Schauder degree of $I-H(\lambda,\cdot)$ with respect to zero, in the set

$$\mathcal{O}:=\bigcup_{s\in[s_n,s'_n]}\{u\in C(\bar\Omega):\|u-s\Phi_1\|_{L^\infty(\partial\Omega)}\le 2\varepsilon s\}.$$

From the homotopy invariance property, $\deg_{LS}(I-H(\lambda,\cdot),\mathcal{O},0)$ is well defined and independent of λ for $\lambda\in[\lambda_{min},\lambda_{max}]$. In particular

$$\deg_{LS}(I-H(\lambda_n,\cdot),\mathcal{O},0)=\deg_{LS}(I-H(\lambda'_n,\cdot),\mathcal{O},0). \tag{3.7}$$

Since from part (i) $\lambda_n<\mu_{1,n}$, the linearized operator $I-\lambda_nS-S[g_s(\lambda_n,x,u_n)\cdot]$ is invertible and consequently u_n is an isolated fixed point. Therefore the fixed point index is well defined and moreover

$$\mathrm{i}\ (H(\lambda_n,\cdot),u_n)=\deg_{LS}(I-\lambda_nS-S[g_s(\lambda_n,x,u_n)\cdot],\mathcal{O},0)=(-1)^{m(\lambda_n)}=1$$

where $m(\lambda_n)$ is sum of the algebraic multiplicities of the eigenvalues of the linearization strictly smaller than λ_n and $m(\lambda_n)=0$ if the linearization has no eigenvalues $\mu_{i,n}$ of this kind.

Moreover, from hypothesis u_n is the only solution in K_n for the value of the parameter $\lambda=\lambda_n$, we deduce $\deg_{LS}(I-H(\lambda_n,\cdot),\mathcal{O},0)=\mathrm{i}\ (H(\lambda_n,\cdot),u_n)$.

On the other side

$$\mathrm{i}\ (H(\lambda'_n,\cdot),u'_n)=\deg_{LS}(I-\lambda'_nS-S[g_s(\lambda_n,x,u_n)\cdot),\mathcal{O},0]=-1$$

and likewise $\deg_{LS}(I-H(\lambda'_n,\cdot),\mathcal{O},0)=\mathrm{i}\ (H(\lambda'_n,\cdot),u'_n)=-1$ which contradicts (3.7) and the proof is accomplished. □

4. **Two examples**

4.1. **The oscillatory nonlinearity.** We try to summarize some of the known results for the nonlinearity (1.6). In [1] it is proved that if $\alpha<1$, for any $\beta\in\mathbb{R}$, and $C\in\mathbb{R}$, there is an unbounded branch of positive solutions, see [1, Theorem 3.4]. Assume from now in advance that $\beta>0$. In [1, Theorem 4.3] it is proved that if $C>1$, then the bifurcation is subcritical while if $C<-1$, the bifurcation is supercritical and in any case there are no resonant solutions, see Figure 2. In [3] it is proved that if $\beta>0$, $\alpha+\beta<1$, and $|C|<1$, there exist unbounded sequences of subcritical and supercritical solutions, subcritical and supercritical turning points and infinite resonant solutions, see Figure 1. Case $|C|=1$ is a critical case. In this particular example, if $|C|=1$ we have an infinite sequence of resonant solutions given by

$$u_k(x):=[(2k\pm 1/2)\pi]^{1/\beta}\Phi_1(x),\qquad k\ge 0.$$

In this paper we proved that if

$$\beta>0,\qquad \alpha+\beta<1,\qquad \text{and}\qquad \forall C,$$

then the unbounded branch of positive solutions contains a sequence of stable solutions, a sequence of unstable solutions and a sequence of turning points, see Theorems 1.2 and 1.3.

Note that if $\alpha+\beta\ge 1$ then $g_s\not\to 0$ as $s\to\infty$ and therefore the eigenvalue of the linearized equation does not converge to the first boundary Steklov eigenvalue, i.e. $\mu_n\not\to\sigma_1$ as $n\to\infty$, see Lemma 3.1. In addition, condition (H3) in Theorem 1.3 cannot be satisfied, and stability of the solutions cannot be deduced from the signs on multiples of the eigenfunction, see the arguments explained at the beginning of Section 3 and also Lemma 3.3. Thus, the restriction $\alpha+\beta<1$ is needed to guarantee both, for the convergence of eigenvalues and eigenvectors to σ_1 and Φ_1 respectively, and for hypothesis (H3) to be satisfied.

4.2. **An example for the case $N = 1$.** We make explicit some ideas on the one dimensional case. We know that the bifurcation problem is a two parameter non linear problem that can be treated using finite dimensional techniques, see section 8 in [1].

Observe that if we consider equation (1.1) in the one dimensional domain $\Omega = (0,1)$, we can rewrite it as

$$\begin{cases} -u_{xx} + u &= 0, & \text{in } (0,1) \\ -u_x(0) &= \lambda u + g(\lambda, 0, u(0)). \\ u_x(1) &= \lambda u + g(\lambda, 1, u(1)), \end{cases}$$

The general solution of the differential equation is $u(x) = ae^x + be^{-x}$ and therefore the nonlinear boundary conditions provides two nonlinear equations in terms of two constants a and b. The function $u = ae^x + be^{-x}$ is a solution if (λ, a, b) satisfy

$$\begin{pmatrix} -(1+\lambda) & (1-\lambda) \\ (1-\lambda)e & -(1+\lambda)e^{-1} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} g(\lambda, 0, a+b) \\ g(\lambda, 1, ae + be^{-1}) \end{pmatrix}$$

In this case we only have two Steklov eigenvalues,

$$\sigma_1 = \frac{e-1}{e+1} < \sigma_2 = \frac{1}{\sigma_1} = \frac{e+1}{e-1}.$$

Choose $g(\lambda, x, s) = s^\alpha \ \sin(s^\beta)$ for any $\alpha < 1,\ \beta > 0$, see Fig 4.

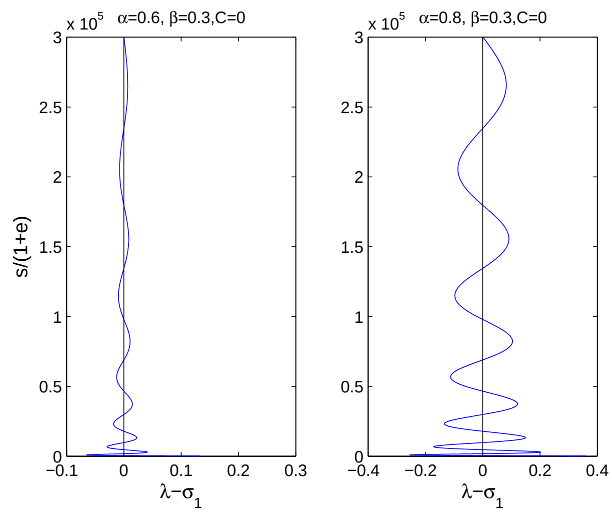

FIGURE 4. A bifurcation diagram of changing stability solutions, on the left $\alpha + \beta < 1$, and on the right $\alpha + \beta > 1$ and in both cases $\lambda \to \sigma_1$.

The eigenvalue of the linearized equation is

$$\mu_1\big(-g_s(\lambda(s), \cdot, u_s)\big) := \frac{e-1}{e+1} - \alpha \frac{\sin\left[[s(e+1)]^\beta\right]}{[s(e+1)]^{1-\alpha}} - [s(e+1)]^{\alpha+\beta-1} \cos\left[[s(e+1)]^\beta\right].$$

If

$$[s(e+1)]^\beta = \begin{cases} (2k+1)\pi \\ \frac{(2k+1)\pi}{2} \\ 2k\pi \end{cases}, \qquad \text{then} \quad \mu_1\big(\lambda(s), u_s\big) - \lambda(s) \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

and we can conclude that (σ_1, u_{2k+1}), where

$$u_{2k+1}(x) := \frac{[(2k+1)\pi]^{1/\beta}}{e+1}(e^x + e^{1-x}) \qquad \text{for any} \quad k \in \mathbb{Z},$$

is a stable solution. Likewise, (σ_1, u_{2k}) is a sequence of unstable solutions where

$$u_{2k}(x) := \frac{(2k\pi)^{1/\beta}}{e+1}(e^x + e^{1-x}) \qquad \text{for any} \quad k \in \mathbb{Z}.$$

Moreover, (λ_k^*, u_k^*) is an unbounded sequence of turning points, where

$$\lambda_k^* := \frac{e-1}{e+1} - \frac{(-1)^k\,\alpha}{[(k+1/2)\pi]^{1-\alpha}}, \qquad u_k^*(x) := \frac{[(2k+1)\pi]^{1/\beta}}{2(e+1)}(e^x + e^{1-x}).$$

The bifurcated branch from infinity contains stable and unstable solutions, and there are an unbounded sequence of turning points. See Figures 1, 2 and 3 for a bifurcation diagram when $N = 1$. In that case, there is not restriction on the size of β, see Fig. 4.

We notice that with respect to the linearization, the things are different depending on $\alpha+\beta$. If $\alpha + \beta \geq 1$ then $\mu_1\big(\lambda(s), \cdot, u_s\big) \not\to \sigma_1$ as $s \to \infty$. On the other side, the eigenvalue of the linearized equation satisfies $\mu_1\big(\lambda(s), \cdot, u_s\big) \to \sigma_1$ as $s \to \infty$, whenever $\alpha + \beta < 1$, see Fig. 5.

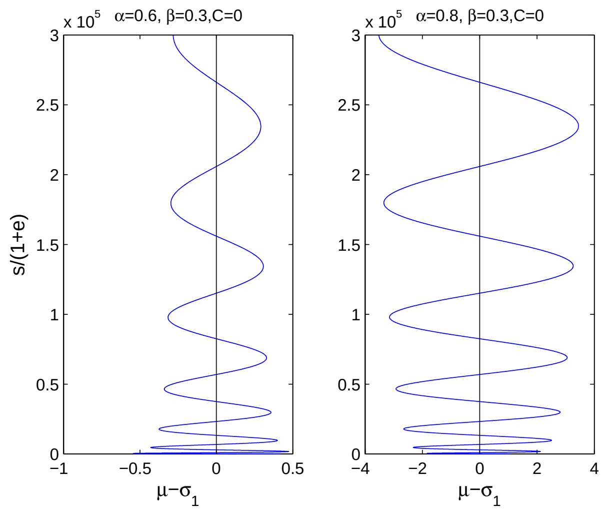

FIGURE 5. The difference between $\mu - \sigma_1$. On the left $\alpha+\beta < 1$, and $\mu \to \sigma_1$, on the right $\alpha + \beta > 1$ and $\mu \not\to \sigma_1$.

Moreover, if $\alpha + \beta < 1$,

$$\begin{aligned}
\underline{\mathbf{F}}_+ \quad &:= \int_{\partial\Omega} \liminf_{s\to+\infty} \frac{sg - s^2 g_s}{|s|^{1+\alpha+\beta}}\,\Phi^{1+\alpha+\beta} \\
&= \int_{\partial\Omega} \liminf_{s\to+\infty}\; -\beta\cos(s^\beta)\;\Phi^{1+\alpha+\beta} = -\beta \int_{\partial\Omega} \Phi^{1+\alpha+\beta}, \\
\overline{\mathbf{F}}_+ \quad &:= \int_{\partial\Omega} \limsup_{s\to+\infty} \frac{sg - s^2 g_s}{|s|^{1+\alpha+\beta}}\,\Phi^{1+\alpha+\beta} \\
&= \int_{\partial\Omega} \limsup_{s\to+\infty}\; -\beta\cos(s^\beta)\;\Phi^{1+\alpha+\beta} = \beta \int_{\partial\Omega} \Phi^{1+\alpha+\beta}
\end{aligned}$$

i.e. $\underline{\mathbf{F}}_+ < 0 < \overline{\mathbf{F}}_+$.

References

[1] Arrieta, J.M., Pardo, R., Rodríguez–Bernal, A. (2007). Bifurcation and stability of equilibria with asymptotically linear boundary conditions at infinity. *Proc. Roy. Soc. of Edinburg* 137 A, no. 2: 225-252.

[2] Arrieta, J.M., Pardo, R., Rodríguez–Bernal, A. (2009). Equilibria and global dynamics of a problem with bifurcation from infinity. *Journal of Differential Equations* 246: 2055-2080.

[3] Arrieta, J.M., Pardo, R., Rodríguez–Bernal, A. Infinite resonant solutions and turning points in a problem with unbounded bifurcation. To appear in *International Journal of Bifurcation and Chaos*.

[4] Cañada, A. (2007). Multiplicity results near the principal eigenvalue for boundary-value problems with periodic nonlinearity, *Math. Nachr* 280, no. 3: 235–241.

[5] Costa, D., Jeggle, H., Schaaf, R., Schmitt, K. (1988). Oscillatory perturbations of linear problems at resonance., *Results Math.* 14, no. 3-4: 275-287.

[6] Galstyan, A., Korman, P., Li, Y. (2006) On the oscillations of the solution curve for a class of semilinear equations., *J. Math. Anal. Appl.* 321: 576-588.

[7] García-Melián, J., Rossi, J. D., Sabina de Lis, J. C. (2009). Existence and uniqueness of positive solutions to elliptic problems with sublinear mixed boundary conditions. *Commun. Contemp. Math.* 11, no. 4: 585–613.

[8] García-Melián, J., Rossi, J. D., Sabina de Lis, J. C. (2009). An elliptic system with bifurcation parameters on the boundary conditions. *Journal of Differential Equations* 247, no. 3: 779-810.

[9] Korman P. (2008). An oscillatory bifurcation from infinity, and from zero., *Nonlinear differ. equ. appl.* 15: 335-345.

[10] Rabinowitz, P. H. (1973). On Bifurcation From Infinity, *J. Differential Equations* 14: 462-475.

Alfonso Castro, Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, USA.
E-mail address: castro@math.hmc.edu

Rosa Pardo, Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040–Madrid, Spain.
E-mail address: rpardo@mat.ucm.es

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2010

1. CONTINUITY OF DYNAMICAL STRUCTURES FOR NON-AUTONOMOUS EVOLUTION EQUATIONS UNDER SINGULAR PERTURBATIONS, J. Arrieta, A. N. Carvalho, J. Langa and A. Rodríguez-Bernal.

2. ON THE LONG TIME BEHAVIOUR OF NON-AUTONOMOUS LOTKA-VOLTERRA MODELS WITH DIFFUSION VIA THE SUB-SUPER TRA JECTORY METHOD, J.A. Langa, A. Rodríguez-Bernal and A. Suárez.

3. MODELLING AND SIMULATION OF A POLLUTED WATER PUMPING PROCESS, C. Alavani, R. Glowinski, S. Gomez, B.Ivorra, P. Joshi and A. M. Ramos.

4. EXPANDING THE ASYMPTOTIC EXPLOSIVE BOUNDARY BEHAVIOR OF LARGE SOLUTIONS TO A SEMILINEAR ELLIPTIC EQUATION, S. Alarcón, G. Díaz, R. Letelier and J. M. Rey.

5. A SINGULAR PERTURBATION IN A LINEAR PARABOLIC EQUATION WITH TERMS CONCENTRATING ON THE BOUNDARY, A. Rodríguez-Bernal.

6. ISOTHERMALISATION FOR A NON-LOCAL HEAT EQUATION, E. Chasseigne and R. Ferreira.

7. HOMOGENIZATION IN A THIN DOMAIN WITH AN OSCILLATORY BOUNDARY, J. M. Arrieta and M. C. Pereira

8. VERY RAPIDLY VARYING BOUNDARIES IN EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS. THE CASE OF A NON UNIFORMLY LIPSCHITZ DEFORMATION, J.M. Arrieta and S. Bruschi

9. PERTURBATION OF ANALYTIC SEMIGROUPS IN SCALES OF BANACH SPACES AND APPLICATIONS TO PARABOLIC EQUATIONS WITH LOW REGULARITY DATA, A. Rodríguez-Bernal

10. IDENTIFICATION OF A PRESSURE DEPENDENT HEAT TRANSFER COEFFICIENT, A. Fraguela, J. A. Infante, Á. M. Ramos and J. M. Rey.

11. MATHEMATICAL MODELING FOR PROTEIN FOLDING DEVICES. APPLICATIONS TO HIGH PRESSURE PROCESSING AND MICROFLUIDIC MIXERS, J. Bello Rivas, J. A. Infante, B. Ivorra, J. López Redondo, P. Martínez Ortigosa, A. M. Ramos, J. M. Rey, and N. Smith

12. A VARIANCE-EXPECTED COMPLIANCE APPROACH FOR TOPOLOGY OPTIMIZATION, M. Carrasco, B. Ivorra, R. Lecaros and A. M. Ramos

13. MODELING AND SIMULATION OF CLASSICAL SWINE FEVER VIRUS SPREAD BETWEEN AND WITHIN FARMS, B.Ivorra, B.Martinez-Lopez, J. M. Sanchez-Vizcaino and A.M. Ramos

14. NONLINEAR PARABOLIC PROBLEMS IN THIN DOMAINS WITH A HIGHLY OSCILLATORY BOUNDARY, J. Arrieta, A. C. Carvalho, M. C. Pereira and R. P. Silva.

15. SINGULAR LIMIT FOR A NONLINEAR PARABOLIC EQUATION WITH TERMS CONCENTRATING ON THE BOUNDARY, A. Jiménez-Casas and A. Rodríguez-Bernal

16. SENSITIVITY ANALYSIS OF A DEFAULT TIME MODEL FOR CREDIT RISK PORTFOLIO MANAGEMENT, R. Abella Muñoz, I. Armero Huertas, B. Ivorra and A. M. Ramos del Olmo

17. OPTIMIZATION OF A PUMPING SHIP TRAJECTORY TO CLEAN OIL CONTAMINATION IN THE OPEN SEA, S. Gómez, B. Ivorra and A. M. Ramos del Olmo

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2011

1. APPROXIMATING TRAVELLING WAVES BY EQUILIBRIA OF NON LOCAL EQUATIONS, J. M. Arrieta, M. López-Fernández and E. Zuazua.

2. INFINITELY MANY STABILITY SWITCHES IN A PROBLEM WITH SUBLINEAR OSCILLATORY BOUNDARY CONDITIONS, A. Castro and R. Pardo