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DISSIPATIVE MECHANISM OF A SEMILINEAR HIGHER ORDER PARABOLIC EQUATION IN $\mathbb{R}^N$

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Abstract. It is known that the concept of dissipativeness is fundamental for understanding the asymptotic behavior of solutions to evolutionary problems. While for the second order equations the dissipativeness mechanism has already been satisfactorily understood, for higher order equations in unbounded domains, this has not been yet fully developed. In this paper we investigate the dissipative mechanism of the semilinear 4-th order parabolic equation $u_t = -\Delta^2 u + f(x,u)$ in $\mathbb{R}^N$ with a nonlinear term $f = f(x,u) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ locally Lipschitz in $u \in \mathbb{R}$ uniformly for $x \in \mathbb{R}^N$. For this we assume the structure condition $f(x,u) \leq C(x)u^2 + D(x)|u|$, $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, where we assume that $0 \leq D$ is in $L^s(\mathbb{R}^N)$, $\max\{1, \frac{2N}{N+4}\} \leq s \leq 2$, and $C \in L^r_U(\mathbb{R}^N)$, $r > \max\{\frac{N}{4}, 1\}$, is such that the solutions of the linear equation $u_t = -\Delta^2 u + Cu$ in $L^2(\mathbb{R}^N)$ decay exponentially as $t \to \infty$.

This and suitable growth conditions allow to prove local existence of solutions in $L^2(\mathbb{R}^N)$ or $H^2(\mathbb{R}^N)$, as in [12]. These growth restrictions are related to the so called critical exponents in these spaces. Then global existence and the existence of a global attractor are proven relying in suitable energy estimates on the solutions.

As shown all along the paper one of the main differences with the case of dissipative reaction diffusion equations stem from the lack of maximum principle. This makes the results less complete in the case of higher order equations, although some similarities with the dissipative mechanisms of the latter are pointed out in the paper.

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1. Introduction

In this article we consider the following Cauchy problem in $\mathbb{R}^N$,

$$\begin{cases} u_t + \Delta^2 u = f(x,u), \quad t>0,\ x\in\mathbb{R}^N, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where the nonlinear term is assumed to be of the general form

$$f(x,u) = g(x) + m(x)u + f_0(x,u),\ x\in\mathbb{R}^N,\ u\in\mathbb{R}, \tag{1.2}$$

for some suitable m, g described below and

$$f_0 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \text{ is locally Lipschitz in } u\in\mathbb{R} \text{ uniformly for } x\in\mathbb{R}^N, \tag{1.3}$$

and

$$f_0(x,0) = 0, \quad \frac{\partial f_0}{\partial u}(x,0) = 0,\ x\in\mathbb{R}^N. \tag{1.4}$$

In some cases, depending on the space in which we solve (1.1), we will also require a growth condition in f_0 of the form

$$|f_0(x,u_1) - f_0(x,u_2)| \le c|u_1-u_2|(1+|u_1|^{\rho-1}+|u_2|^{\rho-1}),\ u_1,u_2\in\mathbb{R}. \tag{1.5}$$

for some $\rho > 1$ and $c > 0$. Note that this class of nonlinear terms includes logistic type nonlinearities of the form

$$f(x,u) = g(x) + m(x)u - u|u|^{\rho-1},\ x\in\mathbb{R}^N, u\in\mathbb{R},\ \rho>1$$

under mild assumptions on $g(x), m(x)$.

Our goal here is to describe some general dissipative mechanism for this equation in a suitable functional setting. In this context "dissipative" refers to the properties that solutions of (1.1) are globally defined and bounded in several norms and moreover that they have a well defined asymptotic behavior.

In recent years the asymptotic behavior of solutions of evolutionary equations in unbounded domains has been studied by many authors and much progress has been achieved specially for the case of reaction diffusion problems. By this we mean that in (1.1), one replaces the term $\Delta^2 u$ by $-\Delta u$. See e.g. in chronological order [6], [15], [19], [23], [21], [14], [2], [4], where different conditions have been given to guarantee that the reaction diffusion equation is dissipative and has a well defined asymptotic behavior in terms of a global attractor, a set which contains all the relevant asymptotic dynamics. Also, note that for the case of reaction diffusion equations a very important tool is the maximum principle. This translates into the comparison principle for solutions of the reaction diffusion problem or, in other words, into the monotonicity of the associated semigroup of solutions. Exploiting this tool it has been shown that, in great generality, when the problem is dissipative, it has extremal equilibria, which are the caps of the attractor; [20], [10], [11].

Recall that in [2] it was described a general mechanism for the dissipativeness of a reaction diffusion problem in $\mathbb{R}^N$ and the existence of a global attractor. It was shown in this reference that both the reaction and the diffusion have to collaborate to produce dissipativeness. If this does not happen, then the linear term in the equation is able to produce unbounded global solutions. This is in sharp contrast with the behavior of solutions in bounded domains where the dissipative character of the nonlinearity is enough, independent of the behavior of the linear term, to produce dissipativity. When this collaboration between reaction and diffusion occurs, then one finds suitable estimates on the solutions in $L^\infty(\mathbb{R}^N)\cap L^p(\mathbb{R}^N)$ and even more, one finds that solutions remain small as $|x|\to\infty$ for large times. Both of these results become crucial to prove that the problem is dissipative. Also for both of them the arguments in [2] rely heavily on the comparison principle.

Hence, the primary goal of the present paper is to investigate the dissipative mechanism of the higher order parabolic equation in $\mathbb{R}^N$, (1.1). We remark that, since (1.1) involves a 4-th order elliptic operator in the main part, the maximum principle is no longer available. Thus, for the analysis of (1.1) we have to rely on some "energy" type estimates of solutions. This is the reason why, although local existence for (1.1) can be done in more general spaces, see [12], the asymptotic behavior of solutions is studied in an $L^2(\mathbb{R}^N)$ or in an $H^2(\mathbb{R}^N)$ setting.

In particular we show that, in a similar way as for reaction diffusion equations, if 4-th order diffusion and reaction collaborate, problem (1.1) is dissipative. This collaboration is reflected in the structure condition

$$uf(x,u)\le C(x)u^2+D(x)|u|,\quad x\in\mathbb{R}^N,\ u\in\mathbb{R}. \tag{1.6}$$

for some suitable functions $C(x)$ and $0\le D(x)$ where $C(x)$ will be assumed such that the solutions of the linear problem

$$\begin{cases} u_t+\Delta^2u=C(x)u,\ t>0,\ x\in\mathbb{R}^N,\\ u(0)=u_0\in L^2(\mathbb{R}^N)\end{cases} \tag{1.7}$$

decay exponentially as $t\to\infty$.

More precisely, for (1.1), we will assume that in (1.6) we have

$$0\le D\in L^s(\mathbb{R}^N),\quad \max\{1,\frac{2N}{N+4}\}\le s\le 2,\quad (\text{and } s>1 \text{ if } N=4) \tag{1.8}$$

and

$$\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|C(x)|^r\,dx<\infty,\ \text{for some } r>\max\{\frac{N}{4},1\}, \tag{1.9}$$

that is, $C\in L^r_U(\mathbb{R}^N)$, where this space is defined, for $1\le r\le\infty$ as

$$L^r_U(\mathbb{R}^N)\stackrel{def}{=}\{\phi\in L^r_{loc}(\mathbb{R}^N):\|\phi\|_{L^r_U(\mathbb{R}^N)}=\sup_{y\in\mathbb{R}^N}\|u\|_{L^r(B(y,1))}<\infty\}$$

(see [3, 18] and note that $L^\infty_U(\mathbb{R}^N):=L^\infty(\mathbb{R}^N)$).

Note that the results in [2] for reaction diffusion equations require (1.6) with different integrability exponents s,r in (1.8) and (1.9) respectively.

With this structure conditions and the decay in (1.7) we will be able to show suitable $L^\infty(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$ estimates on the solutions and prove also that they remain small as $|x|\to\infty$ for large times. So, in a sense, the dissipative mechanism is much similar to the one observed in the reaction diffusion equations, even without the maximum principle. There is a difference however in the results we obtained. For (1.1) dissipativity is obtained under the growth restriction in (1.5)

$$\rho<1+\frac{8}{N-4},\quad \text{for } N\ge 5.$$

For reaction diffusion problems, no such growth restriction is needed. This is again due to the maximum principle, see [2].

In order to make precise the remaining assumptions on (1.1), we will assume that in (1.2) we have

$$m \in L^r_U(\mathbb{R}^N), \quad r > \max\{\frac{N}{4}, 1\} \tag{1.10}$$

and, for simplicity,

$$g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{1.11}$$

Note that from (1.6) we get $|g(x)| \leq D(x)$.

Hence, our goal in this paper is to prove the following results. First, for initial data in $L^2(\mathbb{R}^N)$

Theorem 1.1. *Assume conditions (1.2)–(1.4), (1.10), (1.11) and (1.6)-(1.9) hold, and assume moreover that (1.5) holds with*

$$\rho \leq \rho_c^1 := 1 + \frac{8}{N}.$$

Then for $u_0 \in L^2(\mathbb{R}^N)$ there exits a unique globally defined solution of (1.1). In particular

$$S(t)u_0 = u(t; u_0) \quad t \geq 0$$

defines a strongly continuous semigroup in $L^2(\mathbb{R}^N)$.

Moreover, for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to H^2(\mathbb{R}^N)$ is bounded.

Observe that for $u_0 \in L^2(\mathbb{R}^N)$ local solutions of (1.1) are obtained in Theorem 2.4 with $p = 2$.

On the other hand, for initial data in $H^2(\mathbb{R}^N)$ we have

Theorem 1.2. *Assume conditions (1.2)–(1.4), (1.10), (1.11) and (1.6)-(1.9) hold. If $N \geq 4$ assume moreover that (1.5) holds with*

$$\rho < \rho_c^2 := 1 + \frac{8}{N-4}.$$

Then for $u_0 \in H^2(\mathbb{R}^N)$ there exits a unique globally defined solution of (1.1). In particular

$$S(t)u_0 = u(t; u_0) \quad t \geq 0$$

defines a strongly continuous semigroup in $H^2(\mathbb{R}^N)$.

Observe that for $u_0 \in H^2(\mathbb{R}^N)$ local solutions of (1.1) are obtained in Theorem 2.5 with $p = 2$.

Using this result, we will also show the following result that can be applied to some supercritical cases in $L^2(\mathbb{R}^N)$.

Theorem 1.3. *Assume conditions (1.2)–(1.4), (1.10), (1.11) and assume*

$$\frac{\partial f}{\partial u}(x, u) \leq L(x), \qquad x \in \mathbb{R}^N, u \in \mathbb{R},$$

with $L \in L^{\sigma}_{U}(\mathbb{R}^N)$, $\sigma > \max\{\frac{N}{4}, 1\}$. If $N \geq 4$ assume moreover that (1.5) holds with

$$\rho < \rho_c^2 = 1 + \frac{8}{N-4}.$$

Then
i) Theorem 1.2 applies for initial data in $H^2(\mathbb{R}^N)$. Hence (1.1) defines a semigroup $\{S(t) : t \geq 0\}$ in $H^2(\mathbb{R}^N)$.
ii) The semigroup $\{S(t) : t \geq 0\}$ extends uniquely to a semigroup in $L^2(\mathbb{R}^N)$ and for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to H^2(\mathbb{R}^N)$ is bounded.

Then, in order to obtain, for large times, uniform estimates on all solution we will assume, in addition that solutions of (1.7) decay exponentially as $t \to \infty$. For this we need that (2.3) holds for some $\omega_0 > 0$, see Theorem 2.1 below. Then in the next result we obtain uniform estimates of the solutions in the spaces $H^{\sigma}_{\tau}(\mathbb{R}^N)$, for any $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$, where $x_- = \min\{x, 0\}$ denotes the negative part of $x \in \mathbb{R}$. This coefficient $\beta^*(\tau)$ establishes the maximal regularity we can obtain for the solutions and it comes out from the results in the linear problem in Section 2.

Theorem 1.4. *Assume that solutions of (1.7) decay exponentially as $t \to \infty$.*
A) For the semigroup in $L^2(\mathbb{R}^N)$ in Theorem 1.1, with

$$\rho < \rho_c^1 = 1 + \frac{8}{N},$$

there exist bounded absorbing sets in $L^2(\mathbb{R}^N)$, in $L^{\infty}(\mathbb{R}^N)$ and in $H^{\sigma}_{\tau}(\mathbb{R}^N)$, for any $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$. That is, if X denotes any of the spaces above, there exists an $R_0 > 0$ such that for any bounded set $B \subset L^2(\mathbb{R}^N)$ of initial data of (1.1), there exists $T = T(B)$ such that

$$\|u(t; u_0)\|_X \leq R_0 \quad \textit{for all } t \geq T.$$

In particular, the norm of all equilibria of (1.1) is bounded in X by R_0.
Furthermore, for $t > 0$, the semigroup $S(t)$ is bounded from $L^2(\mathbb{R}^N)$ into X.

B) For the semigroup in $H^2(\mathbb{R}^N)$ in Theorem 1.2:
The orbits of bounded sets in $H^2(\mathbb{R}^N)$ for $\{S(t) : t \geq 0\}$ are almost immediately bounded in $L^{\infty}(\mathbb{R}^N)$ and in $H^{\sigma}_{\tau}(\mathbb{R}^N)$, for any $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$. That is, if X denotes any of the spaces above, for any bounded set $B \subset H^2(\mathbb{R}^N)$ of initial data of (1.1) and for any $\varepsilon > 0$, there exists $K = K(B)$ such that

$$\|u(t; u_0)\|_X \leq K \quad \textit{for all } u_0 \in B \textit{ and } t \geq \varepsilon.$$

There exists a bounded absorbing set in $L^2(\mathbb{R}^N)$. That is, there exists an $R_0 > 0$ such that for any bounded set $B \subset H^2(\mathbb{R}^N)$ of initial data of (1.1), there exists $T = T(B)$ such that

$$\|u(t; u_0)\|_{L^2(\mathbb{R}^N)} \leq R_0 \quad \textit{for all } t \geq T.$$

The set of equilibria of (1.1) is a bounded set in $L^{\infty}(\mathbb{R}^N)$ and in $H^{\sigma}_{\tau}(\mathbb{R}^N)$, for any $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.

C) For the semigroup in $L^2(\mathbb{R}^N)$ in Theorem 1.1 with

$$\rho \leq \rho_c^1 = 1 + \frac{8}{N}$$

or in Theorem 1.3, we have that the orbits of bounded sets in $L^2(\mathbb{R}^N)$ for $\{S(t) : t \geq 0\}$ are almost immediately bounded in $L^\infty(\mathbb{R}^N)$ and in $H^\sigma_\tau(\mathbb{R}^N)$, for any $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$. That is, if X denotes any of the spaces above, for any bounded set $B \subset L^2(\mathbb{R}^N)$ of initial data of (1.1) and for any $\varepsilon > 0$, there exists $K = K(B)$ such that

$$\|u(t; u_0)\|_X \leq K \quad \text{for all } u_0 \in B \text{ and } t \geq \varepsilon.$$

There is a bounded absorbing set in $L^2(\mathbb{R}^N)$. That is, there exist an $R_0 > 0$ such that for any bounded set $B \subset L^2(\mathbb{R}^N)$ of initial data of (1.1), there exists $T = T(B)$ such that

$$\|u(t; u_0)\|_{L^2(\mathbb{R}^N)} \leq R_0 \quad \text{for all } t \geq T.$$

The set of equilibria of (1.1) is a bounded set in $L^\infty(\mathbb{R}^N)$ and in $H^\sigma_\tau(\mathbb{R}^N)$, for any $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.

Finally, we will show that the solutions remain small as $|x| \to \infty$ for large times and hence we prove the following result.

Theorem 1.5. *Under the assumptions Theorem 1.4, the semigroup defined by (1.1) in $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, has a global attractor $\mathbf{A}$, which is compact in $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, invariant $S(t)\mathbf{A} = \mathbf{A}$, and bounded in $H^\sigma_\tau(\mathbb{R}^N)$ for every $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.*

Moreover $\mathbf{A}$ attracts bounded sets of $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, in $H^\sigma_\tau(\mathbb{R}^N)$ for every $2 \leq \tau < \infty$ and $\sigma < 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.

Furthermore

$$\mathbf{A} = \mathcal{W}^u(\mathcal{E}),$$

which is the unstable set of the set $\mathcal{E}$ of equilibria of (1.1).

The paper is organized as follows. In Section 2 we collect some results on the linear equation and on local existence, which are taken from [12].

Then global existence is addressed in Section 3, where in particular we prove Theorems 1.1, 1.2, 1.3 and 1.4.

The existence of the attractor, including the proof of Theorem 1.5 can be found in Section 4.

Finally, some examples are presented in Section 5 while Section 6 includes some comments on the role played by the critical exponent $\rho_c^2 = 1 + \frac{8}{N-4}$ in our results.

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2. Some previous results on linear and nonlinear equations

In this Section we collect some results from [12] on the linear and nonlinear equations. For this, we consider problem (1.1) with initial data in some space of Bessel potentials, which we generically denote $H^{\alpha}_p(\mathbb{R}^N)$, (see [22]). When $p=2$ we will denote these spaces as $H^{\alpha}(\mathbb{R}^N)$ which are Hilbert spaces.

First, regarding the linear problem involving the bi-Laplacian operator in $\mathbb{R}^N$ we have the following result.

Theorem 2.1. *Suppose that $C \in L^r_U(\mathbb{R}^N)$ and $r > \max\{\frac{N}{4}, 1\}$.*
i) Then the operator $A_C = \Delta^2 - C(x)I$ is a sectorial operator in $L^p(\mathbb{R}^N)$ and $-A_C$ generates a C^0 analytic semigroup, $\{e^{-A_C t} : t \geq 0\}$, in $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.
ii) The scale of fractional power spaces, $\{E^{\alpha}_p,\ \alpha \in \mathbb{R}\}$, associated to this operator, is given by

$$E^{\alpha}_p = \begin{cases} H^{4\alpha}_p(\mathbb{R}^N) & \text{for } 0 \leq \alpha \leq \beta^*(p) \leq 1, \\ (H^{-4\alpha}_{p'}(\mathbb{R}^N))' & \text{for } -1 \leq -\beta_*(p) \leq \alpha < 0, \end{cases} \tag{2.1}$$

with $0 < \beta^*(p) = 1 + \left(\frac{N}{4p} - \frac{N}{4r}\right)_- \leq 1$ and $\beta_*(p) = \beta^*(p') = 1 + \left(\frac{N}{4p'} - \frac{N}{4r}\right)_-$, where $x_- = \min\{x, 0\}$ denotes the negative part of $x \in \mathbb{R}$.
iii) On this scale of spaces, the analytic semigroup generated by $-A_C$ satisfies, for some $\omega \in \mathbb{R}$,

$$\|e^{-A_C t}\|_{\mathcal{L}(E^{\sigma}_p, E^{\xi}_p)} \leq M \frac{e^{-\omega t}}{t^{\xi-\sigma}} \quad t > 0,\ -\beta_*(p) \leq \sigma \leq \xi \leq \beta^*(p), \tag{2.2}$$

iv) Also, if $p = 2$ then (2.2) is satisfied for some $\omega > 0$ if and only if there is a certain $\omega_0 > 0$ such that

$$\int_{\mathbb{R}^N} (|\Delta\phi|^2 - C(x)\phi^2) \geq \omega_0 \|\phi\|^2_{L^2(\mathbb{R}^N)}, \tag{2.3}$$

for all $\phi \in H^2(\mathbb{R}^N)$. We say then that the C^0 analytic semigroup $\{e^{-A_C t} : t \geq 0\}$ in $L^2(\mathbb{R}^N)$ is exponentially decaying as $t \to \infty$.

Remark 2.2. *i) Observe that $\beta^*(p) = 1$ iff $r \geq p$ and, for all $1 < p < \infty$,*

$$\beta^*(p) \geq 1 - \frac{N}{4r} > 0.$$

Hence, the interval $[-\beta_*(p), \beta^*(p)]$ contains at least the symmetric interval

$$[-1 + \frac{N}{4r}, 1 - \frac{N}{4r}].$$

Also, the length of the interval $(-\beta_*(p), \beta^*(p))$ is $L = \beta^*(p) + \beta^*(p')$ and then

$$L = \begin{cases} 1 + \beta^*(p'), & \text{if } p' \geq r \geq p \\ 1 + \beta^*(p), & \text{if } p \geq r \geq p' \\ 2 & \text{if } r \geq p, p' \\ 2 + \frac{N}{4} - \frac{N}{2r} & \text{if } p, p' \geq r. \end{cases} \tag{2.4}$$

Note that in any case $L > 1$ since $r > \max\{\frac{N}{4}, 1\}$.

ii) Note that we can use the usual notation

$$H_p^{-4\alpha}(\mathbb{R}^N) = (H_{p'}^{-4\alpha}(\mathbb{R}^N))' \quad \alpha > 0$$

and then (2.1) becomes $E_p^\alpha = H_p^{4\alpha}(\mathbb{R}^N)$ *for* $\alpha \in [-\beta_*(p), \beta^*(p)]$.
iii) Also, the case $r \geq p$ *reflects that the potential is suitable integrable with respect to the base space,* $L^p(\mathbb{R}^N)$. *Hence, in this case the potential can be naturally handled as a perturbation of the bi-Laplacian operator. See for example* [1] *for a similar situation for the case of second order operators.*

On the other hand when $r < p$, *the potential is poorly integrable with respect to the base space and it is more difficult to handle as a perturbation of the bi-Laplacian.*
iv) Note that it is implicit in (2.3) that since C *satisfies (1.9) and* $\phi \in H^2(\mathbb{R}^N)$, *then* $C\phi^2 \in L^1(\mathbb{R}^N)$.

Another useful result on the linear equation is the following, see [12].

Lemma 2.3. *Suppose that* $C \in L^r_U(\mathbb{R}^N)$ *with* $r > \max\{\frac{N}{4}, 1\}$. *Then*
i)The domain of $A_C = \Delta^2 - C(x)I$ *in* $L^2(\mathbb{R}^N)$, $D_{L^2}(A_C)$, *is included in* $H^2(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} A_C\phi\psi = \int_{\mathbb{R}^N} \Delta\phi\Delta\psi - \int_{\mathbb{R}^N} C(x)\phi\psi = \int_{\mathbb{R}^N} \phi A_C\psi, \qquad \phi, \psi \in D_{L^2}(A_C).$$

Furthermore, there exists $\omega_0 \in \mathbb{R}$ *such that*

$$\int_{\mathbb{R}^N} (|\Delta\phi|^2 - C(x)\phi^2) \geq \omega_0 \|\phi\|^2_{L^2(\mathbb{R}^N)} \text{ for each } \phi \in H^2(\mathbb{R}^N).$$

ii) If ω_0 *is as above, there is a continuous decreasing real valued function* $\omega(\nu)$ *defined in a certain interval* $[0, \nu_0]$ *such that for* $\nu \in [0, \nu_0]$,

$$\int_{\mathbb{R}^N} \big((1-\nu)|\Delta\phi|^2 - C(x)\phi^2\big) \geq \omega(\nu) \int_{\mathbb{R}^N} \phi^2 \text{ for all } \phi \in H^2(\mathbb{R}^N),$$

and

$$\lim_{\nu \to 0^+} \omega(\nu) = \omega(0) = \omega_0.$$

Note that in fact the constant ω_0 in the Lemma 2.3 above, gives a lower bound of the bottom spectrum of A_C in $L^2(\mathbb{R}^N)$. So part ii) of the Lemma above is a sort of continuity of the bottom spectrum with respect to the diffusion coefficient.

Then we have the following results on the local existence of (1.1).

Theorem 2.4. *Assume (1.2)–(1.4) with* m *as in (1.10),* $g \in L^p(\mathbb{R}^N)$ *and suppose that (1.5) holds with some*

$$1 < \rho \leq \rho_c^1 := 1 + \frac{4p}{N}$$

for $1 < p < \infty$. *Then (1.1) is locally well posed in* $L^p(\mathbb{R}^N)$.

Now we consider local well posedness in $H^2_p(\mathbb{R}^N)$, $1 < p < \infty$. For this, note that we need that the scale of spaces in (2.1) contains $H^2_p(\mathbb{R}^N)$, which requires

$$\beta^*(p) = 1 + \Big(\frac{N}{4p} - \frac{N}{4r}\Big)_- > \frac{1}{2}, \tag{2.5}$$

that is, $\frac{N}{r} - \frac{N}{p} < 2$, where $x_+ = \max\{x, 0\}$ denotes the positive part of $x \in \mathbb{R}$.

Note that (2.5) is satisfied if $r \geq p$ or if $p \leq \frac{N}{2}$, since $r > \frac{N}{4}$. Also, (2.5) is satisfied for $p = 2$ since $r > \max\{\frac{N}{4}, 1\}$. Hence, we have

Theorem 2.5. *Assume (1.2)–(1.4) and (1.10). Then the problem (1.1) is locally well posed in $H^2_p(\mathbb{R}^N)$, with $1 < p < \infty$, provided that $g \in L^p(\mathbb{R}^N)$ and either*

(i) $2 > \frac{N}{p} > \frac{N}{r} - 2$,

(ii) $2 = \frac{N}{p}$ *and (1.5) holds with some* $1 < \rho < \infty$,

(iii) $2 < \frac{N}{p}$ *and (1.5) holds with some* $1 < \rho \leq \rho_c^2 := 1 + \frac{4p}{N-2p}$.

In both Theorems 2.4 and 2.5 a solution of (1.1) with an initial value $u_0 \in L^p(\mathbb{R}^N)$, or $u_0 \in H^2_p(\mathbb{R}^N)$ respectively, is defined on the maximal interval of existence $[0, \tau_{u_0})$ and satisfies on this interval the variation of constants formula

$$u(t) = e^{(-\Delta^2+mI)t}u_0 + \int_0^t e^{(-\Delta^2+mI)(t-s)}\big(f_0(\cdot, u(s)) + g\big)ds, \tag{2.6}$$

where $e^{(-\Delta^2+mI)t}$ is the semigroup in $L^p(\mathbb{R}^N)$ as in Theorem 2.1.

Concerning further regularity properties of the solution note that if $u_0 \in L^p(\mathbb{R}^N)$

$$u \in C([0, \tau_{u_0}), L^p(\mathbb{R}^N)) \cap C((0, \tau_{u_0}), H_p^{4\beta^*(p)}(\mathbb{R}^N)) \cap C^1((0, \tau_{u_0}), L^p(\mathbb{R}^N)),$$

while if $u_0 \in H^2_p(\mathbb{R}^N)$

$$u \in C([0, \tau_{u_0}), H^2_p(\mathbb{R}^N)) \cap C((0, \tau_{u_0}), H_p^{4\beta^*(p)}(\mathbb{R}^N)) \cap C^1((0, \tau_{u_0}), L^p(\mathbb{R}^N)),$$

where $\beta^*(p) = 1 + \left(\frac{N}{4p} - \frac{N}{4r}\right)_-$ as in Theorem 2.1.

In the "subcritical" cases, that is when $\rho < \rho_c^1$, or $\rho < \rho_c^2$, respectively, the maximal interval of existence of a solution has the property that

$$\tau_{u_0} < \infty \ \text{ implies } \ \limsup_{t \to \tau_{u_0}^-} \|u(t)\|_{L^p(\mathbb{R}^N)} = \infty, \tag{2.7}$$

or respectively,

$$\limsup_{t \to \tau_{u_0}^-} \|u(t)\|_{H^2_p(\mathbb{R}^N)} = \infty \tag{2.8}$$

(see [17, Theorem 3.3.4] and [7, Corollary 1.1]). The critical cases, that is when $\rho = \rho_c^1$, or $\rho = \rho_c^2$, respectively are more involved and (2.7), (2.8) are not true in general; see [7] for related results.

3. Global well posedness

The goal in this section is to show that assuming the structure condition (1.6)–(1.9), that is

$$vf(x, v) \leq C(x)v^2 + D(x)|v|, \quad x \in \mathbb{R}^N, \ v \in \mathbb{R},$$

with C and D as in (1.9) and (1.8), the local solutions of (1.1) constructed in Section 2 with $p = 2$ actually exists for all $t \geq 0$. For this we will restrict to the case $p = 2$ and will denote below by $H^\sigma(\mathbb{R}^N)$ the spaces $H^\sigma_2(\mathbb{R}^N)$. Note that the reason for restricting $p = 2$ is that, since no comparison principle is available for (1.1), we have to rely in suitable "energy" type estimates on the solutions.

We start from the following technical lemma.

Lemma 3.1. *If f_0 satisfies (1.3) and (1.5) then there exists a decomposition*

$$f_0(x,v) = f_{01}(x,v) + f_{02}(x,v),\ x \in \mathbb{R}^N,\ v \in \mathbb{R},$$

where $f_{01}(x,0) = f_{02}(x,0) = 0$,

$$f_{01} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \ \ \textit{is a globally Lipschitz map}$$

and there exists $\hat{c} > 0$ such that for all $v_1, v_2 \in \mathbb{R}$

$$|f_{02}(x,v_1) - f_{02}(x,v_2)| \leq \hat{c}|v_1 - v_2|(|v_1|^{\rho-1} + |v_2|^{\rho-1}). \tag{3.1}$$

Proof: Define

$$f_{01}(x,v) = \begin{cases} f_0(x,v),\ x \in \mathbb{R},\ |v| \leq 1, \\ f_0(x,1),\ x \in \mathbb{R},\ |v| > 1, \end{cases}$$

and

$$f_{02}(x,v) = f_0(x,v) - f_{01}(x,v),\ x \in \mathbb{R}^N,\ v \in \mathbb{R}.$$

With the aid of (1.3), choosing $L_0 > 0$ as a Lipschitz constant for f_0 restricted to $\mathbb{R}^N \times [-1,1]$, we have that

$$|f_{01}(x,v_1) - f_{01}(x,v_2)| \leq L_0|v_1 - v_2|,\ x \in \mathbb{R}^N,\ v_1, v_2 \in \mathbb{R}.$$

Using the above relations and (1.5) we obtain that (3.1) holds. □

In what follows we will make use of the following results that allows us to obtain suitable estimates on the solutions of (1.1). We start with the following simple situation.

Proposition 3.2. *Assume that f satisfies (1.2)–(1.5), (1.10) and assume u is a solution of (1.1) as in Theorem 2.4 or Theorem 2.5.*

If $u \in L^\infty((0,T), L^{s_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ with $T \leq \infty$ and $g \in L^{s_0}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ then for any $s \geq s_0$ and $\sigma \leq 4\beta^*(s) = 4 + \left(\frac{N}{s} - \frac{N}{r}\right)_-$ we have that

$$\|u\|_{L^\infty((\varepsilon,T),H^\sigma_s(\mathbb{R}^N))} \leq L(\varepsilon, \|u\|_{L^\infty((0,T),L^{s_0}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))})$$

for some continuous function $L(\varepsilon,\cdot)$ and any $\varepsilon > 0$ small enough.

Proof: We write (1.1) as

$$u_t + \Delta^2 u - m(x)u + \lambda u = g(x) + \lambda u + f_0(x,u)$$

where $\lambda > 0$ is large enough such that the semigroup generated by $-\Delta^2 + m(x) - \lambda$, which we denote by $T(t)$, decays exponentially.

Since u is bounded in $L^\infty(\mathbb{R}^N)$, we can truncate the nonlinear term f_0 and assume it is globally Lipschitz. In particular f_0 is Lipschitz from $L^s(\mathbb{R}^N)$ into itself. Thus, if $g \in L^s(\mathbb{R}^N)$ then define $h(x,u) = g(x) + \lambda u + f_0(x,u)$ for $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Thus, h is globally Lipschitz and bounded from $L^s(\mathbb{R}^N)$ into itself

Now, for $s \geq s_0$ we have

$$\|h(u(t))\|_{L^\infty((0,T),L^s(\mathbb{R}^N))} \leq \|g\|_{L^s(\mathbb{R}^N)} + \|f_0(u(t))\|_{L^\infty((0,T),L^s(\mathbb{R}^N))} \leq C,$$

where C depends on the norm of u in $L^\infty((0,T),L^s(\mathbb{R}^N))$.

Therefore, if $\beta^*(s)<1$ we get for $t\geq\varepsilon$,

$$\|u(t)\|_{H_s^{4\beta^*(s)}(\mathbb{R}^N)}\leq \frac{ce^{-\omega(t-\varepsilon)}}{(t-\varepsilon)^{\beta^*(s)}}\|u(\varepsilon)\|_{L^s(\mathbb{R}^N)}+\int_\varepsilon^t \frac{ce^{-\omega(t-r)}}{(t-r)^{\beta^*(s)}}\|h(u(r))\|_{L^s(\mathbb{R}^N)}dr$$

which is uniformly bounded for $2\varepsilon\leq t<T$, since $\omega>1$.

If $\beta^*(s)=1$ we can apply, Theorem 5 in [8] to get that the set $\{u(t),t\in(\varepsilon,T)\}$ is a bounded subset of $H^4_s(\mathbb{R}^N)$ and the bound in the norm of the domain will depend on ε and $\|u\|_{L^\infty((\varepsilon,T),L^s(\mathbb{R}^N))}$. □

The following result gives sufficient conditions to obtain an estimate in $L^\infty(\mathbb{R}^N)$.

Proposition 3.3. *Assume that f satisfies (1.2)–(1.5), (1.10) and assume u is a solution of (1.1) as in Theorem 2.4 or Theorem 2.5.*

If $u\in L^\infty((0,T),L^{s_0}(\mathbb{R}^N))$ with $T\leq\infty$ and $g\in L^{s_0}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, with

$$s_0>\frac{N}{4}(\rho-1) \tag{3.2}$$

then

$$\|u\|_{L^\infty((\varepsilon,T),L^\infty(\mathbb{R}^N))}\leq K(\varepsilon,\|u\|_{L^\infty((0,T),L^{s_0}(\mathbb{R}^N))})$$

for some continuous function $K(\varepsilon,\cdot)$ and any $\varepsilon>0$ small enough.

Proof: Using the splitting from Lemma 3.1, we write (1.1) as

$$u_t+\Delta^2u-m(x)u+\lambda u=g(x)+\lambda u+f_{01}(x,u)+f_{02}(x,u)$$

where $\lambda>0$ is large enough such that the semigroup generated by $-\Delta^2+m(x)-\lambda$, which we denote by $T(t)$, decays exponentially.

Thus, if $g\in L^s(\mathbb{R}^N)$ then define $h_1(x,u)=g(x)+\lambda u+f_{01}(x,u)$ and $h_2(x,u)=f_{0,2}(x,u)$ for $(x,u)\in\mathbb{R}^N\times\mathbb{R}$. Thus, h_1 is globally Lipschitz and bounded from $L^s(\mathbb{R}^N)$ into itself and $\|h_2(u)\|_{L^s(\mathbb{R}^N)}\leq c\|u\|^\rho_{L^{s\rho}(\mathbb{R}^N)}$ for every $s>1$.

Assume $u\in L^\infty((0,T),L^s(\mathbb{R}^N))$ for some $s\geq s_0$. Denote $\alpha=\beta^*(s)$ if $\beta^*(s)<1$, while $\alpha<1$ and close to 1, if $\beta^*(s)=1$. Then we show below that

$$\|u\|_{L^\infty((2\varepsilon,T),H_s^{4\alpha}(\mathbb{R}^N))}\leq L(\varepsilon,\|u\|_{L^\infty((\varepsilon,T),L^s(\mathbb{R}^N))}) \tag{3.3}$$

for some continuous function $L(\varepsilon,z)$ and $\varepsilon>0$ small enough.

Assume for a while that (3.3) holds true. Then, using now the embeddings of Bessel potential spaces we get a bound of u in $L^\infty((2\varepsilon,T),L^\infty(\mathbb{R}^N))$ provided $4\alpha-N/s>0$. Note that this is always satisfied if $\beta^*(s)<1$, since $4\beta^*(s)-N/s=4-N/r>0$, or if $\beta^*(s)=1$ and $s>\frac{N}{4}$.

On the other hand, if $4\alpha-N/s<0$, we get a bound of u in $L^\infty((2\varepsilon,T),L^{\tilde{s}}(\mathbb{R}^N))$, where $4\alpha-N/s=-N/\tilde{s}$.

Now the argument can be iterated as, by assumption, $g\in L^{\tilde{s}}(\mathbb{R}^N)$. In particular we have that from a bound in $L^\infty((\varepsilon,T),L^s(\mathbb{R}^N))$ we obtain a bound in $L^\infty((2\varepsilon,T),L^{\tilde{s}}(\mathbb{R}^N))$.

In fact if we measure the gain of regularity in the last step above, the relationship between s and $\tilde{s}$, neglecting in both of them an arbitrarily small term, can be expressed as $-N/\tilde{s}=4-N/s$ or, in terms of the variable $w=-N/s$, $-N/\tilde{s}:=H(w)=4+w$.

Note that, starting at $w_0 = -N/s_0$, we have to iterate $H(w)$, which is an increasing function until in a finite number of steps we reach some value $w_{k+1} = H(w_k) > 0$ such that $\tilde{s}_{k+1} > \frac{N}{4}$. Since $H(w)$ is an increasing function with no fixed points, this is always possible.

Therefore it remain to prove (3.3), which will be done in two steps.

Step 1. Estimate for short times. Assume $2\varepsilon < t^* \leq T$ and $t^* \leq 1$. From the variations of constants formula (2.6), we have

$$u(t) = T(t-\varepsilon)u(\varepsilon) + \int_\varepsilon^t T(t-s)h_1(u(r))\,dr + \int_\varepsilon^t T(t-s)h_2(u(r))\,dr, \quad t \in (\varepsilon, t^*).$$

Hence, denoting below by c different constants that do not depend on u, we have using Theorem 2.1, for some $\omega > 0$,

$$\|u(t)\|_{H_s^{4\alpha}(\mathbb{R}^N)} \leq \frac{ce^{-\omega(t-\varepsilon)}}{(t-\varepsilon)^\alpha}\|u(\varepsilon)\|_{L^s(\mathbb{R}^N)} + c\int_\varepsilon^t \frac{e^{-\omega(t-r)}}{(t-r)^\alpha}\big(\|g\|_{L^s(\mathbb{R}^N)} + \|u(r)\|_{L^s(\mathbb{R}^N)}\big)\,dr$$
$$+ c\int_\varepsilon^t \frac{e^{-\omega(t-r)}}{(t-r)^{\alpha-\zeta}}\|u^\rho(r)\|_{H_s^{4\zeta}(\mathbb{R}^N)}\,dr \quad t \in (\varepsilon, t^*),$$

for some $-\beta_*(s) \leq \zeta \leq 0$ with $\alpha - \zeta < 1$, to be chosen below. Using the embedding

$$L^q(\mathbb{R}^N) \hookrightarrow H_s^{4\zeta}(\mathbb{R}^N)$$

for some $4\zeta - \frac{N}{s} \leq -\frac{N}{q} \leq -\frac{N}{s}$, we have

$$c\int_\varepsilon^t \frac{e^{-\omega(t-r)}}{(t-r)^{\alpha-\zeta}}\|u^\rho(r)\|_{H_s^{4\zeta}(\mathbb{R}^N)}\,dr \leq c\int_\varepsilon^t \frac{e^{-\omega(t-r)}}{(t-r)^{\alpha-\zeta}}\|u(r)\|^\rho_{L^{q\rho}(\mathbb{R}^N)}\,dr.$$

In this last term we are going to use the Nirenberg-Gagliardo inequality and $\omega > 0$ to obtain the bound

$$c\int_\varepsilon^t \frac{1}{(t-r)^{\alpha-\zeta}}\|u(r)\|^{\theta\rho}_{H_s^{4\alpha}(\mathbb{R}^N)}\|u(r)\|^{(1-\theta)\rho}_{L^s(\mathbb{R}^N)}\,dr.$$

For this, we need

$$-\frac{N}{s} \leq -\frac{N}{q\rho} < 4\alpha - \frac{N}{s}$$

and finding $\theta \in (0,1)$ such that

$$-\frac{N}{s} \leq -\frac{N}{q\rho} \leq (4\alpha - \frac{N}{s})\theta - (1-\theta)\frac{N}{s} = 4\alpha\theta - \frac{N}{s}$$

which we write as

$$-\frac{N\rho}{s} \leq -\frac{N}{q} \leq 4\alpha\rho\theta - \frac{N\rho}{s}.$$

Now we show that this is possible even with

$$\theta\rho \in (0,1).$$

In fact the choice of ζ, q, θ as above is possible provided

$$\max\{4\zeta - \frac{N}{s}, -\frac{N\rho}{s}\} \leq -\frac{N}{q} < 4\alpha - \frac{N\rho}{s}, \tag{3.4}$$

with $\zeta > \alpha - 1$ and $\zeta \geq -\beta_*(s)$.

Hence, if $\beta^*(s)=1$ we have $\alpha<1$ and close to 1 and then the lower bound on ζ reduces to $\zeta>\alpha-1$

$$4\alpha-4-\frac{N}{s}<4\zeta-\frac{N}{s}\le-\frac{N}{q}<4\alpha-\frac{N\rho}{s},$$

which is satisfied iff $-4-\frac{N}{s}<-\frac{N\rho}{s}$ which, in turn, is satisfied because of (3.2) and $s\ge s_0$. Also, $-\frac{N\rho}{s}<4\alpha-\frac{N\rho}{s}$ is clearly satisfied, and (3.4) holds in this case.

On the other hand, if $\beta^*(s)<1$ we have $\alpha=\beta^*(s)$ and then

$$4\beta^*(s)-4-\frac{N}{s}<4\zeta-\frac{N}{s}\le-\frac{N}{q}<4\beta^*(s)-\frac{N\rho}{s},$$

which is satisfied iff $-4-\frac{N}{s}<-\frac{N\rho}{s}$ which, in turn, is satisfied because of (3.2) and $s\ge s_0$. Also,

$$-4\beta_*(s)-\frac{N}{s}<4\zeta-\frac{N}{s}\le-\frac{N}{q}<4\beta^*(s)-\frac{N\rho}{s},$$

which is satisfied iff

$$\frac{N}{4s}(\rho-1)<\beta^*(s)+\beta_*(s).$$

Note that from (2.4), $\beta^*(s)+\beta_*(s)$ is greater than 1. Finally, $-\frac{N\rho}{s}<4\beta^*(s)-\frac{N\rho}{s}$ is also satisfied and (3.4) holds in this case too.

Then, using the Nirenberg-Gagliardo inequality, and $(t-\varepsilon)^\alpha\le 1$, $\omega>0$, we get

$$\begin{aligned}(t-\varepsilon)^\alpha\|u(t)\|_{H^{4\alpha}_s(\mathbb{R}^N)}&\le c\big(\|u(\varepsilon)\|_{L^s(\mathbb{R}^N)}+\|g\|_{L^s(\mathbb{R}^N)}+\|u\|_{L^\infty((\varepsilon,T),L^s(\mathbb{R}^N))}\big)\\&\quad+c(t-\varepsilon)^\alpha\int_\varepsilon^t\frac{1}{(t-r)^{\alpha-\zeta}}\|u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}\|u(r)\|^{(1-\theta)\rho}_{L^s(\mathbb{R}^N)}\,dr\\&\le a+b(t-\varepsilon)^\alpha\int_\varepsilon^t\frac{1}{(t-r)^{\alpha-\zeta}}\|u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}\,dr,\quad t\in(\varepsilon,t^*)\end{aligned}$$

where

$$a:=c\big(\|g\|_{L^s(\mathbb{R}^N)}+2\|u\|_{L^\infty((\varepsilon,T),L^s(\mathbb{R}^N))}\big),\qquad b:=c\|u\|^{(1-\theta)\rho}_{L^\infty((\varepsilon,T),L^s(\mathbb{R}^N))}.$$

Now observe that

$$\begin{aligned}&(t-\varepsilon)^\alpha\int_\varepsilon^t\frac{1}{(t-r)^{\alpha-\zeta}}\|u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}\,dr\\&\qquad\le\sup_{r\in(\varepsilon,t]}\|(r-\varepsilon)^\alpha u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}(t-\varepsilon)^\alpha\int_\varepsilon^t\frac{1}{(t-r)^{\alpha-\zeta}(r-\varepsilon)^{\alpha\theta\rho}}\,dr\end{aligned}$$

and using the change of variable $r=\varepsilon+z(t-\varepsilon)$ and using $(t-\varepsilon)^{1-\alpha\theta\rho}\le 1$, we get

$$(t-\varepsilon)^\alpha\int_\varepsilon^t\frac{1}{(t-r)^{\alpha-\zeta}}\|u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}\,dr\le B(1-\alpha+\zeta,1-\theta\rho\alpha)\sup_{r\in(\varepsilon,t]}\|(r-\varepsilon)^\alpha u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}$$

where $B(\cdot,\cdot)$ is Euler's Beta function.

Thus we have

$$(t-\varepsilon)^\alpha\|u(t)\|_{H^{4\alpha}_s(\mathbb{R}^N)}\le a+bB(1-\alpha+\zeta,1-\theta\rho\alpha)\sup_{r\in(\varepsilon,t]}\|(r-\varepsilon)^\alpha u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)},\quad t\in(\varepsilon,t^*).$$

Hence,

$$z(t) := \sup_{r\in(\varepsilon,t]} \|(r-\varepsilon)^{\alpha} u(r)\|_{H^{4\alpha}_s(\mathbb{R}^N)}, \quad t\in(\varepsilon,t^*),$$

satisfies

$$0 \le z(t) \le a + bB(1-\alpha+\zeta, 1-\theta\rho\alpha) z(t)^{\theta\rho},$$

and, since $\rho\theta<1$, we get

$$0\le z(t)\le z_0,$$

where z_0 is the unique positive root of $z = a + bB(1-\alpha+\zeta,1-\theta\rho\alpha)z^{\theta\rho}$. We have then

$$\|u(t)\|_{H^{4\alpha}_s(\mathbb{R}^N)} \le \frac{z_0}{(t-\varepsilon)^{\alpha}},\ \varepsilon < t\le t^*,$$

which completes the proof of Step 1. In particular, $\|u(2\varepsilon)\|_{H^{4\alpha}_s(\mathbb{R}^N)} \le \frac{z_0}{\varepsilon^{\alpha}}$, which will be used in Step 2.

Step 2. Estimate for longer times. As in Step 1, from the variations of constants formula, now for times $2\varepsilon < t < T \le \infty$, we have

$$\|u(t)\|_{H^{4\alpha}_s(\mathbb{R}^N)} \le ce^{-\omega(t-2\varepsilon)}\|u(2\varepsilon)\|_{H^{4\alpha}_s(\mathbb{R}^N)} + c\int_{2\varepsilon}^{t} \frac{e^{-\omega(t-r)}}{(t-r)^{\alpha}}\big(\|g\|_{L^s(\mathbb{R}^N)} + \|u(r)\|_{L^s(\mathbb{R}^N)}\big)\,dr$$
$$+\, c\int_{2\varepsilon}^{t} \frac{e^{-\omega(t-r)}}{(t-r)^{\alpha-\zeta}} \|u(r)\|^{\rho}_{L^{q\rho}(\mathbb{R}^N)}\,dr, \quad t\in(2\varepsilon,T).$$

Using again the Nirenberg-Gagliardo inequality we get now

$$\begin{aligned}\|u(t)\|_{H^{4\alpha}_s(\mathbb{R}^N)} &\le c\big(\|u(2\varepsilon)\|_{H^{4\alpha}_s(\mathbb{R}^N)} + \|g\|_{L^s(\mathbb{R}^N)} + \|u\|_{L^{\infty}((2\varepsilon,T),L^s(\mathbb{R}^N))}\big)\\ &\quad + c\int_{2\varepsilon}^{t}\frac{e^{-\omega(t-r)}}{(t-r)^{\alpha-\zeta}}\|u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}\|u(r)\|^{(1-\theta)\rho}_{L^s(\mathbb{R}^N)}\,dr\\ &\le \hat a + \hat b\int_{2\varepsilon}^{t}\frac{e^{-\omega(t-r)}}{(t-r)^{\alpha-\zeta}}\|u(r)\|^{\theta\rho}_{H^{4\alpha}_s(\mathbb{R}^N)}\,dr, \quad t\in(2\varepsilon,T),\end{aligned}$$

where, using the bound in Step 1,

$$\hat a := c\big(\varepsilon^{-\alpha}z_0 + \|g\|_{L^s(\mathbb{R}^N)} + \|u\|_{L^{\infty}((2\varepsilon,T),L^s(\mathbb{R}^N))}\big) \qquad \hat b := c\|u\|^{(1-\theta)\rho}_{L^{\infty}((2\varepsilon,T),L^s(\mathbb{R}^N))}.$$

Letting now

$$\hat z(t) := \sup_{r\in(2\varepsilon,t]}\|u(r)\|_{H^{4\alpha}_s(\mathbb{R}^N)}, \quad t\in(2\varepsilon,T),$$

we obtain that $\hat z(t)$ satisfies

$$0\le \hat z(t)\le \hat a + \hat b\frac{\Gamma(1-\alpha+\zeta)}{\omega^{1-\alpha+\zeta}}\hat z^{\theta\rho}(t),$$

and thus, since $\theta\rho<1$,

$$0\le \hat z(t)\le \hat z_0,$$

where $\hat z_0$ is the unique positive root of $\hat z = \hat a + \hat b\frac{\Gamma(1-\alpha+\zeta)}{\omega^{1-\alpha+\zeta}}\hat z^{\theta\rho}$. Therefore,

$$\|u(t)\|_{H^{4\alpha}_s(\mathbb{R}^N)} \le \hat z_0, \quad 2\varepsilon < t < T,$$

which completes the proof of (3.3). □

Remark 3.4. *Observe that, (3.2) implies that* $1+\frac{4s_0}{N} > \rho$*. Therefore, according to Theorem 2.4, (3.2) can be read as (1.1) being subcritical in* $L^{s_0}(\mathbb{R}^N)$*.*

Now we prove the following result, whose first part applies to any local solution of (1.1) which lies in $L^2(\mathbb{R}^N)$. Note that this solution could be a local solution with initial data in $L^2(\mathbb{R}^N)$, as in Theorem 2.4, with $p = 2$ or a local solution with initial data in $H^2_p(\mathbb{R}^N)$ as in Theorem 2.5, with $p > 1$ such that $H^2_p(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$, that is $\frac{2N}{N+4} \le p \le 2$.

Proposition 3.5. *Assume (1.6)–(1.9), that is*

$$vf(x, v) \le C(x)v^2 + D(x)|v|, \quad x \in \mathbb{R}^N,\ v \in \mathbb{R}.$$

with C *and* D *as in (1.9) and (1.8). Then*
i) If $u(t)$ *is a local solution in* $L^2(\mathbb{R}^N)$*, the norm of* $u(t)$ *in* $L^2(\mathbb{R}^N)$ *does not blow–up in finite time.*
ii) Additionally, assume conditions (1.2)–(1.5), (1.10), (1.11) and

$$\rho < 1 + \frac{8}{N} = \rho^1_c \tag{3.5}$$

then the local solutions of (1.1) for $u_0 \in L^2(\mathbb{R}^N)$ *as in Theorem 2.4, with* $p = 2$*, are globally defined. In particular*

$$S(t)u_0 = u(t; u_0) \quad t \ge 0$$

defines a strongly continuous semigroup in $L^2(\mathbb{R}^N)$*.*

Proof: Multiplying (1.1) by u, using (1.6), (1.8) and the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)} &\le \int_{\mathbb{R}^N} C(x)u^2 + \int_{\mathbb{R}^N} D(x)|u| \\ &\le \int_{\mathbb{R}^N} C(x)u^2 + \|D\|_{L^s(\mathbb{R}^N)}\|u\|_{H^2_2(\mathbb{R}^N)}.\end{aligned}$$

Taking into account that the norm $\|\Delta u\|_{L^2(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}$ is equivalent to the norm in $H^2(\mathbb{R}^N)$ and using Cauchy inequality we get, for any $\varepsilon > 0$,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)} \le \int_{\mathbb{R}^N} C(x)u^2 + \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} + \frac{\varepsilon}{2}\big(\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \|u\|^2_{L^2(\mathbb{R}^N)}\big).$$

Rewriting the above inequality as

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{\varepsilon}{2}\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \Big((1-\varepsilon)\|\Delta u\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} C(x)u^2\Big) \le \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} + \frac{\varepsilon}{2}\|u\|^2_{L^2(\mathbb{R}^N)}$$

and using Lemma 2.3 we get

$$\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^N)} + \varepsilon\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \omega\int_{\mathbb{R}^N} u^2 \le \frac{2c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} \tag{3.6}$$

for some $\omega \in \mathbb{R}$. Thus Gronwall's lemma gives the bound of the $L^2(\mathbb{R}^N)$ norm of u on finite time intervals, which proves i).

For ii) note that since ρ satisfies (3.5), then (2.7) implies that the solutions are global. □

Now if we assume some more dissipativity we get

Theorem 3.6. *Under the assumptions of Proposition 3.5 assume furthermore that (2.3) holds for some $\omega_0 > 0$.*

Then there exist bounded absorbing sets in $L^2(\mathbb{R}^N)$, in $L^\infty(\mathbb{R}^N)$ and in $H^\sigma_\tau(\mathbb{R}^N)$, for any $2 \le \tau < \infty$ and $\sigma \le 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$, for the semigroup of solutions in $L^2(\mathbb{R}^N)$ defined in part ii) of Proposition 3.5. That is, if X denotes any of the spaces above, there exists an $R_0 > 0$ such that for any bounded set $B \subset L^2(\mathbb{R}^N)$ of initial data of (1.1), there exists $T = T(B)$ such that

$$\|u(t; u_0)\|_X \le R_0 \quad \textit{for all } t \ge T.$$

In particular, the norm of all equilibria of (1.1) is bounded in X by R_0.

Furthermore, for $t > 0$, the semigroup $S(t)$ is bounded from $L^2(\mathbb{R}^N)$ into X.

Proof: Proceeding as in Proposition 3.5 and using Lemma 2.3 with sufficiently small $\varepsilon > 0$, we get (3.6) with some $\omega > 0$. Gronwall's Lemma now implies that

$$\|u\|^2_{L^2(\mathbb{R}^N)} \le \|u_0\|^2_{L^2(\mathbb{R}^N)} e^{-\omega t} + \frac{2c}{\varepsilon\omega}\|D\|^2_{L^s(\mathbb{R}^N)}, \ t \ge 0, \tag{3.7}$$

which gives the result in $L^2(\mathbb{R}^N)$. Then, Proposition 3.3 with $s_0 = 2$ and Proposition 3.2 give the result in the other spaces. The rest is immediate. □

Now we turn into some estimates in $H^2(\mathbb{R}^N)$ of the solutions. For this the arguments are based in using the functional

$$E(\phi) := \frac{1}{2}\int_{\mathbb{R}^N} |\Delta\phi|^2 - \int_{\mathbb{R}^N} F(x,\phi), \ \phi \in H^2(\mathbb{R}^N) \tag{3.8}$$

with $F(x,u) = \int_0^u f(x,s)\,ds$, which will be shown below to be a suitable energy functional for (1.1) in $H^2(\mathbb{R}^N)$.

First observe that from (1.2) and (1.4) we get $F(x,u) = g(x)u + m(x)\frac{u^2}{2} + F_0(x,u)$. If (1.5) is satisfied then

$$|F_0(x,u)| \le c(u^2 + |u|^{\rho+1}).$$

Hence, if $N \le 3$, or if $N \ge 4$ and $\rho < \rho_c^2 = 1 + \frac{8}{N-4}$ then $F_0(\cdot,u) \in L^1(\mathbb{R}^N)$. Also from part iv) in Remark 2.2 we have that $mu^2 \in L^1(\mathbb{R}^N)$. Therefore E is well defined on $H^2(\mathbb{R}^N)$.

Lemma 3.7. *Assume $u(t)$ is a local solution in $H^2(\mathbb{R}^N)$. Then as long as it exists, it satisfies the estimates*

$$E(u(t;u_0)) \le E(u(s;u_0)) \quad t \ge s$$

$$\int_s^t \|u_t(r)\|^2_{L^2(\mathbb{R}^N)}\,dr + E(u(t;u_0)) = E(u(s;u_0)) \quad t \ge s.$$

Proof: Multiplying (1.1) by u_t we obtain

$$\frac{d}{dt}E(u(t)) = -\|u_t(t)\|^2_{L^2(\mathbb{R}^N)} \le 0. \quad \square$$

We now prove that the energy is bounded below.

Lemma 3.8. *Suppose that the conditions (1.6)–(1.9) hold.*

Then there are positive constants c_1, c_3 and $c_2 \in \mathbb{R}$ such that

$$c_1 \int_{\mathbb{R}^N} |\Delta\phi|^2 + c_2 \int_{\mathbb{R}^N} |\phi|^2 - c_3 \leq E(\phi), \quad \phi \in H^2(\mathbb{R}^N).$$

If in addition, (2.3) holds for some $\omega_0 > 0$, then

$$a_1 \|\phi\|^2_{H^2(\mathbb{R}^N)} - a_2 \leq E(\phi), \quad \phi \in H^2(\mathbb{R}^N).$$

for some positive constants a_1, a_2.

Proof: By assumption on the nonlinear term we have

$$F(x,u) = \int_0^u f(x,s)\,ds \leq \frac{1}{2} C(x) u^2 + D(x)|u|, \ u \in \mathbb{R}$$

and hence

$$2E(\phi) \geq \int_{\mathbb{R}^N} |\Delta\phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - 2\int_{\mathbb{R}^N} D(x)|\phi|, \ \phi \in H^2(\mathbb{R}^N).$$

Using (1.8), Hölder's inequality, the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$ and taking into account that the norm $\|\Delta\phi\|_{L^2(\mathbb{R}^N)} + \|\phi\|_{L^2(\mathbb{R}^N)}$ is equivalent to the $H^2(\mathbb{R}^N)$ norm we get

$$\begin{aligned} 2E(\phi) &\geq \int_{\mathbb{R}^N} |\Delta\phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - 2\|D\|_{L^s(\mathbb{R}^N)} \|\phi\|_{L^{s'}(\mathbb{R}^N)} \\ &\geq \int_{\mathbb{R}^N} |\Delta\phi|^2 - \int_{\mathbb{R}^N} C(x)\phi^2 - \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} - \frac{\varepsilon}{2}(\|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} + \|\phi\|^2_{L^2(\mathbb{R}^N)}), \end{aligned}$$

which we rewrite as

$$2E(\phi) \geq \frac{\varepsilon}{2}\|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \big((1-\varepsilon)|\Delta\phi|^2 - C(x)\phi^2\big) - \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)} - \frac{\varepsilon}{2}\|\phi\|^2_{L^2(\mathbb{R}^N)}.$$

Now using Lemma 2.3 with $\varepsilon > 0$ small enough we obtain

$$2E(\phi) \geq \frac{\varepsilon}{2}\|\Delta\phi\|^2_{L^2(\mathbb{R}^N)} + \omega\|\phi\|^2_{L^2(\mathbb{R}^N)} - \frac{c}{\varepsilon}\|D\|^2_{L^s(\mathbb{R}^N)}$$

with some $\omega \in \mathbb{R}$ which is positive if (2.3) holds for some $\omega_0 > 0$.

In this case, in particular, the solutions of (1.1) are global and define a nonlinear semigroup in $H^2(\mathbb{R}^N)$. Namely, the following result holds.

Theorem 3.9. *Assume conditions (1.2)–(1.4), (1.10), (1.11) and (1.6)-(1.9) hold, that is*

$$vf(x,v) \leq C(x)v^2 + D(x)|v|, \quad x \in \mathbb{R}^N, \ v \in \mathbb{R}.$$

with C and D as in (1.9) and (1.8).

If $N \geq 4$ assume moreover that (1.5) holds with

$$\rho < \rho_c^2 = 1 + \frac{8}{N-4}.$$

i) Then the local solutions of (1.1) for $u_0 \in H^2(\mathbb{R}^N)$ as in Theorem 2.5 with $p = 2$ are globally defined. In particular

$$S(t)u_0 = u(t; u_0) \quad t \geq 0$$

defines a strongly continuous semigroup in $H^2(\mathbb{R}^N)$.

Assume, in addition, that (2.3) holds for some $\omega_0 > 0$. *Then*
ii) The orbits of bounded sets in $H^2(\mathbb{R}^N)$ *for* $\{S(t) : t \geq 0\}$ *are almost immediately bounded in* $L^\infty(\mathbb{R}^N)$ *and in* $H^\sigma_\tau(\mathbb{R}^N)$, *for any* $2 \leq \tau < \infty$ *and* $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$. *That is, if* X *denotes any of the spaces above, for any bounded set* $B \subset H^2(\mathbb{R}^N)$ *of initial data of (1.1) and any* $\varepsilon > 0$, *there exists* $K = K(B)$ *such that*

$$\|u(t;u_0)\|_X \leq K \quad \textit{for all } u_0 \in B \textit{ and } t \geq \varepsilon.$$

iii) There is a bounded absorbing set in $L^2(\mathbb{R}^N)$. *That is, there exist an* $R_0 > 0$ *such that for any bounded set* $B \subset H^2(\mathbb{R}^N)$ *of initial data of (1.1), there exists* $T = T(B)$ *such that*

$$\|u(t;u_0)\|_{L^2(\mathbb{R}^N)} \leq R_0 \quad \textit{for all } t \geq T.$$

iv) The set of equilibria of (1.1) is a bounded set in $L^\infty(\mathbb{R}^N)$ *and in* $H^\sigma_\tau(\mathbb{R}^N)$, *for any* $2 \leq \tau < \infty$ *and* $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.

Proof: From Lemmas 3.7 and 3.8 we have, for $t \geq 0$,

$$c_1 \int_{\mathbb{R}^N} |\Delta u(t;u_0)|^2 + c_2 \int_{\mathbb{R}^N} |u(t;u_0)|^2 - c_3 \leq E(u(t;u_0)) \leq E(u_0)$$

with $c_1, c_3 > 0$ while from part i) of Proposition 3.5 we get a bound in finite time of the $L^2(\mathbb{R}^N)$ norm. Hence, we have bounds in finite time of the $H^2(\mathbb{R}^N)$ norm and since $\rho < \rho_c^2 = 1 + \frac{8}{N-4}$ we get i) (see (2.8)).

If in addition, (2.3) holds for some $\omega_0 > 0$ from Lemma 3.8

$$a_1 \|u(t;u_0)\|^2_{H^2(\mathbb{R}^N)} - a_2 \leq E(u(t;u_0)) \leq E(u_0)$$

with $a_1, a_2 > 0$, and then orbits of bounded sets in $H^2(\mathbb{R}^N)$ are bounded in $H^2(\mathbb{R}^N)$.

If $N \leq 3$ this already implies a bound for u in $L^\infty(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then since $g \in L^\tau(\mathbb{R}^N)$ for any $2 \leq \tau < \infty$ Proposition 3.2 gives the result for all $2 \leq \tau < \infty$.

If $N \geq 5$ we have $H^2(\mathbb{R}^N) \subset L^{\frac{2N}{N-4}}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Hence, Proposition 3.3 with any $2 \leq s_0 \leq \frac{2N}{N-4}$ satisfying (3.2), gives $L^\infty(\mathbb{R}^N)$-bound on the solutions. Note that we can use Proposition 3.3 with s_0 as above because $\rho < \rho_c^2$. Again, since $g \in L^\tau(\mathbb{R}^N)$ for any $2 \leq \tau < \infty$ and we now have $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$-bound on the solutions, Proposition 3.2 gives the result for all $2 \leq \tau < \infty$.

If $N = 4$ we proceed as above with any $s_0 < \infty$ satisfying (3.2).

Note that part iii) follows from (3.6)–(3.7) which still hold.

Finally, note that from (3.6), for any equilibrium of (1.1) we get

$$\varepsilon \|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \omega \|u\|^2_{L^2(\mathbb{R}^N)} \leq \frac{2c}{\varepsilon} \|D\|^2_{L^s(\mathbb{R}^N)}$$

with $\omega > 0$, which gives the bound in $H^2(\mathbb{R}^N)$. Again Propositions 3.3 and 3.2 as above gives iv). □

Remark 3.10. *Note that if (2.3) holds for some* $\omega_0 > 0$ *then the energy in (3.8) is a Lyapunov function for the semigroup* $\{S(t) : t \geq 0\}$ *in* $H^2(\mathbb{R}^N)$, *in the sense of* [16, pp. 49-50]*; that is,*

(i) *$E : H^2(\mathbb{R}^N) \to \mathbb{R}$ is bounded below,*

(ii) *$E(u_0) \to \infty$ as $\|u_0\|_{H^2(\mathbb{R}^N)} \to \infty$,*

(iii) *$E(S(t)u_0)$ is nonincreasing in t for each $u_0 \in H^2(\mathbb{R}^N)$,*

(iv) *if u_0 is such that $S(t)u_0$ is defined for all $t \in \mathbb{R}$ and $E(S(t)u_0) = E(u_0)$ for $t \in \mathbb{R}$ then u_0 is an equilibrium point.*

With this, to conclude the existence of a global attractor it is enough to establish a suitable dissipativeness and asymptotic compactness property of the solutions (see [16]*). This will be our main concern in the next section.*

To conclude this section, observe that combining the arguments in Proposition 3.5 and in Theorem 3.9 we get the following result for the critical case in $L^2(\mathbb{R}^N)$.

Theorem 3.11. *Assume conditions (1.2)–(1.4), (1.10), (1.11) and (1.6)-(1.9) hold, that is*

$$vf(x, v) \leq C(x)v^2 + D(x)|v|, \quad x \in \mathbb{R}^N,\ v \in \mathbb{R}.$$

with C and D as in (1.9) and (1.8), and assume moreover that (1.5) holds with

$$\rho = \rho_c^1 = 1 + \frac{8}{N}.$$

Then the local solutions of (1.1) for $u_0 \in L^2(\mathbb{R}^N)$ as in Theorem 2.4 with $p = 2$ are globally defined. In particular

$$S(t)u_0 = u(t; u_0) \quad t \geq 0$$

defines a strongly continuous semigroup in $L^2(\mathbb{R}^N)$.

Moreover, for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to H^2(\mathbb{R}^N)$ is bounded.

Proof: Consider a local solution for $u_0 \in L^2(\mathbb{R}^N)$ as in Theorem 2.4 with $p = 2$. Then (3.6) implies that for any $0 < t_1 < t_2 < \infty$ such that the solution exist in $[0, t_2]$ we have

$$\int_{t_1}^{t_2} \|u\|^2_{H^2(\mathbb{R}^N)} \leq C(t_2, \|u_0\|_{L^2(\mathbb{R}^N)}).$$

Hence the mean value theorem for the integral implies that there exists some intermediate time $0 < t_1 < \delta < t_2 < \infty$ such that

$$\|u(\delta)\|^2_{H^2(\mathbb{R}^N)} \leq \frac{C(t_2, \|u_0\|_{L^2(\mathbb{R}^N)})}{t_2 - t_1}.$$

Now, we can apply part i) in Theorem 3.9, to get that the solution is global. For this note that for $N \geq 4$ we have $\rho_c^1 = 1 + \frac{8}{N} < \rho_c^2 = 1 + \frac{8}{N-4}$. In particular note that at $t = t_2$ we get a bound on $\|u(t_2)\|^2_{H^2(\mathbb{R}^N)}$ depending on $\|u_0\|_{L^2(\mathbb{R}^N)}$ and t_2 itself.

Note now that again (3.6) implies that for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is bounded, and the argument above proves that for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to H^2(\mathbb{R}^N)$ is bounded. □

Now we show another situation in which (1.1) defines a semigroup in $L^2(\mathbb{R}^N)$, even for some supercritical cases in $L^2(\mathbb{R}^N)$. For this, we will assume a one side monotonicity condition of the form

$$\frac{\partial f}{\partial u}(x, u) \leq L(x), \qquad x \in \mathbb{R}^N, u \in \mathbb{R}, \tag{3.9}$$

with a certain

$$L \in L^{\sigma}_{U}(\mathbb{R}^N), \ \sigma > \max\{\frac{N}{4}, 1\}. \tag{3.10}$$

Namely, the following result holds.

Theorem 3.12. *Assume conditions (1.2)–(1.4), (1.10), (1.11) and assume (3.9) and (3.10). If $N \geq 4$ assume moreover that (1.5) holds with*

$$\rho < \rho^2_c = 1 + \frac{8}{N-4}.$$

Then
i) Part i) of Theorem 3.9 applies for initial data in $H^2(\mathbb{R}^N)$. Hence (1.1) defines a semigroup $\{S(t) : t \geq 0\}$ in $H^2(\mathbb{R}^N)$.
ii) The semigroup $\{S(t) : t \geq 0\}$ extends uniquely to a semigroup in $L^2(\mathbb{R}^N)$ and for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to H^2(\mathbb{R}^N)$ is bounded.

Proof: Note that from (3.9) we have

$$vf(x, v) \leq L(x)v^2 + |g(x)||v|, \ x \in \mathbb{R}^N, \ v \in \mathbb{R},$$

that is (1.6) holds with $C(x) = L(x)$ and $D(x) = |g(x)|$, which satisfy (1.9) and (1.8). Therefore part i) of Theorem 3.9 applies and we get i).

Now take $u^1_0, u^2_0 \in H^2(\mathbb{R}^N)$ and set $z(t) := S(t)u^1_0 - S(t)u^2_0$ for $t \geq 0$. Then from (1.1)

$$z_t + \Delta^2 z = f(x, S(t)u^1_0) - f(x, S(t)u^2_0), \ t > 0.$$

Multiplying this equation by z, integrating over $\mathbb{R}^N$ and using (3.9), we have

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(\mathbb{R}^N)} + \|\Delta z\|^2_{L^2(\mathbb{R}^N)} \leq \int_{\mathbb{R}^N} L(x)|z|^2.$$

Finally, applying Lemma 2.3, we obtain for small enough $\varepsilon > 0$

$$\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2(\mathbb{R}^N)} + \varepsilon\|\Delta z\|^2_{L^2(\mathbb{R}^N)} + \omega\|z\|^2_{L^2(\mathbb{R}^N)} \leq 0 \tag{3.11}$$

with a certain $\omega \in \mathbb{R}$. Thus for any $T > 0$,

$$\|S(t)u^1_0 - S(t)u^2_0\|_{L^2(\mathbb{R}^N)} \leq c(T)\|u^1_0 - u^2_0\|_{L^2(\mathbb{R}^N)}, \ 0 \leq t \leq T$$

and

$$\|S(\cdot)u^1_0 - S(\cdot)u^2_0\|_{L^2((0,T),H^2(\mathbb{R}^N))} \leq c(T)\|u^1_0 - u^2_0\|_{L^2(\mathbb{R}^N)}, \ 0 \leq t \leq T.$$

Thus, the semigroup can be extended uniquely to $L^2(\mathbb{R}^N)$.

Proceeding now as in the proof of Theorem 3.11 we get that for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to H^2(\mathbb{R}^N)$ is bounded. □

Remark 3.13. *i) Note that this last result applies for example when*

$$f(x, v) = m(x)v - v|v|^{\rho-1}, \ x \in \mathbb{R}^N, v \in \mathbb{R},$$

even with

$$\rho^1_c = 1 + \frac{8}{N} < \rho$$

and, if $N \geq 5$,

$$\rho < \rho_c^2 = 1 + \frac{8}{N-4}$$

which is supercritical in $L^2(\mathbb{R}^N)$*, although subcritical in* $H^2(\mathbb{R}^N)$*.*
ii) Note that a very similar argument is carried out for reaction diffusion equations in [2]*. However in that paper no upper bound on* ρ *is need for any dimension* N*. This is due to the fact that for reaction diffusion equations, due to the maximum principle, condition (3.9) implies that solutions corresponding to very smooth initial data are globally defined. Here we rely on part i) in Theorem 3.9 to obtain global solutions for* $H^2(\mathbb{R}^N)$ *instead.*

For both cases of the semigroup constructed in Theorem 3.11 and Theorem 3.12, we have

Theorem 3.14. *Assume conditions (1.2)–(1.4), (1.10), (1.11) and (1.6)-(1.9) hold, that is*

$$vf(x,v) \leq C(x)v^2 + D(x)|v|, \quad x \in \mathbb{R}^N, \ v \in \mathbb{R}.$$

with C *and* D *as in (1.9) and (1.8).*

Assume, in addition that (2.3) holds for some $\omega_0 > 0$*.*

Then for both the semigroups constructed in Theorems 3.11 and 3.12, we have
i) The orbits of bounded sets in $L^2(\mathbb{R}^N)$ *for* $\{S(t) : t \geq 0\}$ *are almost immediately bounded in* $L^\infty(\mathbb{R}^N)$ *and in* $H^\sigma_\tau(\mathbb{R}^N)$*, for any* $2 \leq \tau < \infty$ *and* $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$*. That is, if* X *denotes any of the spaces above, for any bounded set* $B \subset L^2(\mathbb{R}^N)$ *of initial data of (1.1) and any* $\varepsilon > 0$*, there exists* $K = K(B)$ *such that*

$$\|u(t;u_0)\|_X \leq K \quad \textit{for all } u_0 \in B \textit{ and } t \geq \varepsilon.$$

ii) There is a bounded absorbing set in $L^2(\mathbb{R}^N)$*. That is, there exist an* $R_0 > 0$ *such that for any bounded set* $B \subset L^2(\mathbb{R}^N)$ *of initial data of (1.1), there exists* $T = T(B)$ *such that*

$$\|u(t;u_0)\|_{L^2(\mathbb{R}^N)} \leq R_0 \quad \textit{for all } t \geq T.$$

iii) The set of equilibria of (1.1) is a bounded set in $L^\infty(\mathbb{R}^N)$ *and in* $H^\sigma_\tau(\mathbb{R}^N)$*, for any* $2 \leq \tau < \infty$ *and* $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$*.*

Proof: In both cases of Theorems 3.11 and 3.12 the result follows from Theorem 3.9, since for any $t > 0$, $S(t) : L^2(\mathbb{R}^N) \to H^2(\mathbb{R}^N)$ is bounded. □

4. Long time behavior of the solutions

In this section we investigate the dissipative mechanism for (1.1) and prove the existence of a global attractor. For this, after either Theorem 3.6, Theorem 3.14 or Theorem 3.9 it remains to show that the semigroup defined by (1.1) in $L^2(\mathbb{R}^N)$, or $H^2(\mathbb{R}^N)$ respectively, is asymptotically compact, see [16].

First we will show the asymptotic compactness in $L^\tau(\mathbb{R}^N)$ for any $2 \leq \tau < \infty$. For this, we will use a suitable cut-off function as in [23] (see also [2, 13]), to show that the solutions of (1.1) become small "as $|x| \to \infty$" uniformly for large values of time and for bounded sets of initial data.

After this is done we will use the variation of constants formula to prove that this property actually holds in $H^\sigma_\tau(\mathbb{R}^N)$ for every $2 \le \tau < \infty$ and $\sigma < 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.

Lemma 4.1. *Under the assumptions of either Theorem 3.6, Theorem 3.14 or Theorem 3.9, for each B bounded in $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, and for arbitrarily chosen $\varepsilon > 0$ there exist certain $t_0 > 0$ and $R > 0$ such that*

$$\sup_{u_0 \in B} \sup_{t \ge t_0} \|u(t; u_0)\|_{L^\tau(\{|x|>R\})} < \varepsilon \tag{4.1}$$

for any $2 \le \tau < \infty$.

Proof: Note that from Theorem 3.6, Theorem 3.14 or Theorem 3.9 we have bounds of the orbit of B in $H^2(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$. In particular it is enough to prove the result for $\tau = 2$.

Chose any smooth function $\theta_0 : [0, \infty) \to [0, 1]$ such that $\theta_0(z) = 0$ for $z \in [0, 1]$ and $\theta_0(z) = 1$ for $z \ge 2$. Let $\theta(z) = \theta_0^4(z)$ for $z \ge 0$ and define

$$\phi_k(x) = \theta\Big(\frac{|x|^2}{k^2}\Big), \ x \in \mathbb{R}^N, \ k = 1, 2, \dots .$$

Multiplying (1.1) by $u\phi_k$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} u^2\phi_k + \int_{\mathbb{R}^N} |\Delta u|^2\phi_k &= -\int_{\mathbb{R}^N} \Delta u(u\Delta\phi_k - 2\nabla\phi_k \cdot \nabla u) + \int_{\mathbb{R}^N} uf(x,u)\phi_k \\ &=: J_1 + J_2. \end{aligned} \tag{4.2}$$

Now, as a consequence of the bounds in $H^2(\mathbb{R}^N)$ and properties of ϕ_k there exists a constant $c > 0$ such that

$$J_1 = -\int_{\mathbb{R}^N} \Delta u(u\Delta\phi_k - 2\nabla\phi_k \cdot \nabla u) \le \frac{c}{k}.$$

Also, since

$$\Delta(u\phi_k^{\frac{1}{2}}) = \phi_k^{\frac{1}{2}}\Delta u + 2\nabla u \cdot \nabla(\phi_k^{\frac{1}{2}}) + u\Delta(\phi_k^{\frac{1}{2}}),$$

then using again the bounds in $H^2(\mathbb{R}^N)$, we obtain that there is a certain constant, which we again denote by c, such that

$$\begin{aligned} \int_{\mathbb{R}^N} |\Delta u|^2\phi_k &= \int_{\mathbb{R}^N} \big(\Delta(u\phi_k^{\frac{1}{2}}) - 2\nabla u \cdot \nabla(\phi_k^{\frac{1}{2}}) - u\Delta(\phi_k^{\frac{1}{2}})\big)^2 \\ &\ge \int_{\mathbb{R}^N} |\Delta(u\phi_k^{\frac{1}{2}})|^2 - \frac{c}{k}. \end{aligned}$$

On the other hand, applying (1.6), we get

$$J_2 = \int_{\mathbb{R}^N} uf(x,u)\phi_k \le \int_{\mathbb{R}^N} C(x)(u\phi_k^{\frac{1}{2}})^2 + \int_{\mathbb{R}^N} D(x)|u|\phi_k.$$

Hence, (4.2) transforms into

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} u^2\phi_k + \int_{\mathbb{R}^N} |\Delta(u\phi_k^{\frac{1}{2}})|^2 \le \int_{\mathbb{R}^N} C(x)(u\phi_k^{\frac{1}{2}})^2 + \int_{\mathbb{R}^N} D(x)|u|\phi_k + \frac{2c}{k}. \tag{4.3}$$

We next estimate the integral $\int_{R^N} D(x)|u|\phi_k$ by $\|D\phi_k^{\frac{1}{2}}\|_{L^s(\mathbb{R}^N)}\|u\phi_k^{\frac{1}{2}}\|_{L^{s'}(\mathbb{R}^N)}$, and from (1.8) after using the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{s'}(\mathbb{R}^N)$ and an equivalent norm in $H^2(\mathbb{R}^N)$, this can be bounded by $\tilde{c}\|D\phi_k^{\frac{1}{2}}\|_{L^s(\mathbb{R}^N)}(\|\Delta(u\phi_k^{\frac{1}{2}})\|_{L^2(\mathbb{R}^N)} + \|u\phi_k^{\frac{1}{2}}\|_{L^2(\mathbb{R}^N)})$. Hence we get

$$\int_{\mathbb{R}^N} D(x)|u|\phi_k \leq \frac{\delta}{2}(\|\Delta(u\phi_k^{\frac{1}{2}})\|^2_{L^2(\mathbb{R}^N)} + \|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)}) + \frac{\tilde{c}^2}{\delta}\|D\phi_k^{\frac{1}{2}}\|^2_{L^s(\mathbb{R}^N)}. \tag{4.4}$$

From (4.3)–(4.4), we have for $\delta > 0$

$$\frac{1}{2}\frac{d}{dt}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} + \frac{\delta}{2}\|\Delta(u\phi_k^{\frac{1}{2}})\|^2_{L^2(\mathbb{R}^N)} + \Big((1-\delta)\|\Delta(u\phi_k^{\frac{1}{2}})\|^2_{L^2(\mathbb{R}^N)} - \int_{\mathbb{R}^N} C(x)(u\phi_k^{\frac{1}{2}})^2\Big) \leq$$
$$\leq \frac{\delta}{2}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} + \frac{\tilde{c}^2}{\delta}\|D\phi_k^{\frac{1}{2}}\|^2_{L^s(\mathbb{R}^N)} + \frac{2c}{k}$$

and using Lemma 2.3 with $\delta > 0$ small enough we infer that

$$\frac{1}{2}\frac{d}{dt}\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} + \frac{\delta}{2}\|\Delta(u\phi_k^{\frac{1}{2}})\|^2_{L^2(\mathbb{R}^N)} + \omega\|u\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)} \leq \frac{\tilde{c}^2}{\delta}\|D\phi_k^{\frac{1}{2}}\|^2_{L^s(\mathbb{R}^N)} + \frac{2c}{k}$$

with $\omega > 0$. We thus observe that

$$z_k(t) := \|u(t)\phi_k^{\frac{1}{2}}\|^2_{L^2(\mathbb{R}^N)}$$

satisfies the differential inequality

$$z_k' + \omega z_k \leq c_k,$$

with $\omega > 0$, where

$$c_k = \frac{4c}{k} + \frac{2\tilde{c}^2}{\delta}\Big(\int_{\{|x|>k\}} D^s\Big)^{\frac{2}{s}} \to 0 \quad \text{as } k \to \infty.$$

Thus (4.1) follows now using Gronwall's inequality. □

Lemma 4.2. *Under the assumption of Lemma 4.1 each sequence of the form $\{S(t_n)u_{0n}\}$, where $\{u_{0n}\}$ is bounded in $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, and $t_n \to \infty$, has a subsequence convergent in $L^\tau(\mathbb{R}^N)$ for any $2 \leq \tau < \infty$.*

Proof: Given any $\varepsilon > 0$, we know from Lemma 4.1 that there is a certain $k_0, \hat{N} \in \mathbb{N}$ such that

$$\|S(t_n)u_{0n} - S(t_m)u_{0m}\|_{L^\tau(\{|x|>k_0\})} \leq \varepsilon \ \text{ for all } n, m \geq \hat{N}.$$

From the bounds in either Proposition 3.6 or Theorem 3.9, $\{S(t_n)u_{0n}\}$ is bounded in, say, $H_\tau^{4-\frac{N}{r}}(\mathbb{R}^N)$, for any $2 \leq \tau < \infty$, there exists a subsequence, denoted the same, which converges in $L^\tau(\{|x| < k_0\})$. Hence

$$\|S(t_n)u_{0n} - S(t_m)u_{0m}\|_{L^\tau(\mathbb{R}^N)} \leq \varepsilon \ \text{ for all } n, m \geq \tilde{N}.$$

Hence, $\{S(t_n)u_{0n}\}$ is a Cauchy sequence in $L^\tau(\mathbb{R}^N)$, which completes the proof. □

Lemma 4.3. *Under the assumption of Lemma 4.1 each sequence of the form $\{S(t_n)u_{0n}\}$, where $\{u_{0n}\}$ is bounded in $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, and $t_n \to \infty$, has a subsequence convergent in $H^\sigma_\tau(\mathbb{R}^N)$ for any $2 \leq \tau < \infty$ and $\sigma < 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.*

Proof: As a consequence of Lemma 4.2 there exists a subsequence $\{S(t_{n_k}-1)u_{0n_k}\}$ of $\{S(t_n-1)u_{0n}\}$ which is a Cauchy sequence in $Y = L^\tau(\mathbb{R}^N)$. In addition, by either Theorem 3.6, Theorem 3.14 or Theorem 3.9, the orbit of the set $\{u_{0n}\}$ (and thus also the set $\{S(t_n - 1)u_{0n}\}$) is bounded, for $t \geq 1$, in $H^\sigma_\tau(\mathbb{R}^N)$ and in $L^\infty(\mathbb{R}^N)$.

Then we can truncate the nonlinear term in (1.1) and assume that it is Lipschitz from $Z = H^\sigma_\tau(\mathbb{R}^N)$ into $Y = L^\tau(\mathbb{R}^N)$. Then, using the variations of constants formula, from [9, Theorem 3.2.1] we get that $S(1)$ takes $\{S(t_n-1)u_{0n}\}$ into a precompact subset of Z, so that $\{S(t_{n_k})u_{n_k})\}$ has a subsequence convergent in $H^\sigma_\tau(\mathbb{R}^N)$. □

Therefore we get the following

Theorem 4.4. *Under the assumptions of either Theorem 3.6, Theorem 3.14 or Theorem 3.9, the semigroup defined by (1.1) in $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, has a global attractor* $\mathbf{A}$*, which is compact in $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, invariant $S(t)\mathbf{A} = \mathbf{A}$, and bounded in $H^\sigma_\tau(\mathbb{R}^N)$ for every $2 \leq \tau < \infty$ and $\sigma \leq 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.*

Moreover $\mathbf{A}$ *attracts bounded sets of $L^2(\mathbb{R}^N)$, or in $H^2(\mathbb{R}^N)$ respectively, in $H^\sigma_\tau(\mathbb{R}^N)$ for every $2 \leq \tau < \infty$ and $\sigma < 4\beta^*(\tau) = 4 + \left(\frac{N}{\tau} - \frac{N}{r}\right)_-$.*

Furthermore

$$\mathbf{A} = \mathcal{W}^u(\mathcal{E}),$$

which is the unstable set of the set $\mathcal{E}$ of equilibria of (1.1).

Proof: In the case of the semigroup in, $H^2(\mathbb{R}^N)$, by Theorem 3.9, the set of equilibria is bounded in $H^2(\mathbb{R}^N)$ and the orbit of bounded sets is bounded in $H^2(\mathbb{R}^N)$. By Lemma 4.3 the semigroup is asymptotically compact in $H^2(\mathbb{R}^N)$. Since the energy is a Lyapunov functional for (1.1), then the existence of the attractor $\mathbf{A}$, as in the statement, follows; see [16, Theorem 3.8.5, page 51].

For the case of the semigroup in, $L^2(\mathbb{R}^N)$, by Theorem 3.6 or Theorem 3.14 there is a bounded absorbing set in $L^2(\mathbb{R}^N)$, while by Lemma 4.2, the semigroup is asymptotically compact in $L^2(\mathbb{R}^N)$. Again, by [16], the existence of $\mathbf{A}$ follows. But again Theorem 3.6 or Theorem 3.11 imply that all solutions enter $H^2(\mathbb{R}^N)$ and then the attractor is the unstable set of equilibria.

The rest is immediate. □

Remark 4.5. *Note that the result above implies in particular that the attractor of problem (1.1) attracts solutions uniformly in $\mathbb{R}^N$ and even more in $C^{3,\alpha}_b(\mathbb{R}^N)$ for $\alpha < 1$ when $r = \infty$.*

5. Some examples

In this section we show some sample nonlinearities which can be handled with the previous results.

Example 5.1. *Assume*

$$f(x, v) = g(x) - cv + f_0(v), \ x \in \mathbb{R}^N, \ v \in \mathbb{R},$$

with

$$c > 0, \quad vf_0(v) \leq 0, \quad g \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

It is immediate that (1.6) holds with $C(x) = -c$, $D(x) = |g(x)|$ and that the solutions of the linear problem (1.7) are exponentially decaying as $t \to \infty$.

Consequently, Theorem 1.1, 1.2, 1.4 and 1.5 apply. Note that in these results the growth restrictions should be fulfilled as stated in the assumptions of these theorems. Namely, $\rho \leq \rho_c^1$ in the $L^2(\mathbb{R}^N)$-setting, whereas $\rho < \rho_c^2$ for space dimensions $N \geq 5$ in the $H^2(\mathbb{R}^N)$-setting.

If moreover $f_0'(s) \leq L$ then Theorem 1.3 applies and the semigroup can be then extended to $L^2(\mathbb{R}^N)$ even in some supercritical case $\rho_1^c < \rho$ (see Remark 3.13).

Now, observe that if $g = 0$ then the attractor reduces to $\{0\}$, since (3.6) holds with $D = 0$ and $\omega > 0$.

On the other hand, if $g \neq 0$ and $f_0'(s) \leq 0$ the attractor reduces to the unique equilibria, since (3.11) holds with $\omega > 0$.

Example 5.2. *Assume now that*

$$f(x,v) = m(x)v - n(x)|v|^{\rho-1}v, \; x \in \mathbb{R}^N, \; v \in \mathbb{R},$$

where $0 \leq n \in L^\infty(\mathbb{R}^N)$ and

$$1 < \rho < \rho_c^2 = 1 + \frac{8}{N-4} \quad \textit{when } N \geq 4.$$

Observe that $\frac{\partial f}{\partial u}(x,u) \leq m(x)$ and assume (1.10). Hence Theorem 3.12 applies and we have a semigroup of global solutions of (1.1) in $L^2(\mathbb{R}^N)$ and $H^2(\mathbb{R}^N)$. In fact, the semigroup operators are bounded from $L^2(\mathbb{R}^N)$ into $H^2(\mathbb{R}^N)$ for every $t > 0$.

Then to apply Theorems 3.14 and 4.4, we will require that

$$m(x) = m_1(x) + m_2(x), \; x \in \mathbb{R}^N,$$

where

$$m_1, m_2 \in L^r_U(\mathbb{R}^N) \;\; \textit{for} \;\; \max\{1, \frac{N}{4}\} < r \leq \infty$$

and assume that the semigroup generated by $-\Delta^2 + m_1 I$ is exponentially decaying as $t \to \infty$.

Note that with the aid of Young's inequality we now have

$$vf(x,v) = m(x)v^2 - n(x)|v|^{\rho+1} \leq m_1(x)v^2 + C_\rho |m_2(x)|^{\frac{\rho}{\rho-1}} [n(x)]^{\frac{1}{1-\rho}} |v|$$

for each $v \in \mathbb{R}$ and every $x \in \mathbb{R}^N$ such that $n(x) \neq 0$. Thus (1.6) holds with $C(x) = m_1(x)$, $D(x) = \frac{\rho-1}{\rho}|m_2(x)|^{\frac{\rho}{\rho-1}}[n(x)]^{\frac{1}{1-\rho}}$ and Theorems 3.14 and 4.4 apply provided that we also have

$$D = |m_2|^{\frac{\rho}{\rho-1}} n^{\frac{1}{1-\rho}} \in L^s(\mathbb{R}^N) \;\; \textit{for some} \;\; \max\{\frac{2N}{N+4}, 1\} \leq s \leq 2 \quad (\textit{and } s > 1 \textit{ if } N = 4).$$

Observe that in a similar way we can deal with a more general nonlinearities; for example

$$f(x,v) = m(x)v + l(x)p(v) - n(x)v|v|^{\rho-1}, \; x \in \mathbb{R}^N, \; v \in \mathbb{R},$$

where $0 \leq n \in L^\infty(\mathbb{R}^N)$ and $p(v)$ is a polynomial of degree less than ρ.

6. Some comments on critical and supercritical cases

In this section we make some remarks on the role played by the critical exponents $\rho_c^1 = 1 + \frac{8}{N}$ for the $L^2(\mathbb{R}^N)$ setting and $\rho_c^2 = 1 + \frac{8}{N-4}$ for the $H^2(\mathbb{R}^N)$ setting.

First, note that as proved in [12], the critical exponents above appear naturally when proving local existence of (1.1). These exponents are related to the growth of the nonlinear terms only. In particular they do not distinguish nonlinearities that, for large values of u, behave as $\pm|u|^{\rho-1}u$. However one may expect that the two different signs behave different when studying the asymptotic behavior of solutions.

For example for reaction diffusion problems (see discussion in the Introduction) it is shown in [2] that for $f(u) \sim -|u|^{\rho-1}u$, the problem is well posed and dissipative for any value of ρ. On the other hand, for $f(u) \sim |u|^{\rho-1}u$, local existence holds only for $\rho \leq 1 + \frac{4}{N}$ in $L^2(\mathbb{R}^N)$ and $\rho \leq 1 + \frac{4}{N-2}$ in $H^2(\mathbb{R}^N)$. It was actually proved in [5] that in this latter case, if $\rho > 1 + \frac{4}{N}$, or $\rho > 1 + \frac{4}{N-2}$ respectively, the problem is ill posed. Both results in [2] and [5] use again in an essential manner the maximum principle. In particular a key point in [2] is that, comparison principles, allow to obtain bounds in $L^\infty(\mathbb{R}^N)$ of the solution for any value of ρ.

Back to (1.1) observe that for logistic type nonlinearities $f(x,u) = g(x) + m(x)u - u|u|^{\rho-1}$, see Section 5, the energy (3.8) would give under suitable assumption on $g(x), m(x)$, at least formally, bounds on the $H^2(\mathbb{R}^N)$ and the $L^{\rho+1}(\mathbb{R}^N)$ of the solution at any positive time. However note that Proposition 3.3 can only be used with $s_0 = \rho+1$ again if $\rho < \rho_c^2 = 1 + \frac{8}{N-4}$.

As observed below (2.8), for the critical case in $H^2(\mathbb{R}^N)$ it is not enough to bound the $H^2(\mathbb{R}^N)$ norm of the solution to guarantee that it is global. However the critical case in $L^2(\mathbb{R}^N)$ in Theorem 3.11, or even the supercritical cases in Theorem 3.12 can be handled because the solutions enter $H^2(\mathbb{R}^N)$ and in this space they are in a subcritical regime. Hence the bounds in $H^2(\mathbb{R}^N)$ prove the solutions in $L^2(\mathbb{R}^N)$ are global.

However we have not found any good estimate on the solution that would allow us to go beyond the critical exponent in the $H^2(\mathbb{R}^N)$ setting. Here the lack of maximum principle becomes a major difference with reaction diffusion equations.

References

[1] H. Amann, M. Hieber, G. Simonnett, Bounded H^∞ calculus for elliptic operators Differential Integral Equations 7, 653 –613 (1994).

[2] J. M. Arrieta, J. W. Cholewa, T. Dlotko, A. Rodriguez-Bernal, Asymptotic behavior and attractors for reaction diffusion equations in unbounded domains, *Nonlinear Analysis TMA* 56, 515-554 (2004).

[3] J. M. Arrieta, J. W. Cholewa, T. Dlotko, A. Rodriguez-Bernal, Linear parabolic equations in locally uniform spaces, *Math. Models Methods Appl. Sci.* 14, 253-294 (2004).

[4] J. M. Arrieta, J. W. Cholewa, T. Dlotko, A. Rodriguez-Bernal, Dissipative parabolic equations in locally uniform spaces, *Math. Nachr.* 280, 1643-1663 (2007).

[5] J.M. Arrieta, A.Rodríguez–Bernal, "Non well posedness of parabolic equations with supercritical nonlinearities", Comm. Contemporary Math. 6, 733-764 (2004).

[6] A.V. Babin, M.I. Vishik, Attractors of partial differential evolution equations in unbounded domain, *Proc. Roy. Soc. Edinburgh* 116A, 221-243 (1990).

[7] A. N. Carvalho, J. W. Cholewa, Continuation and asymptotics of solutions to semilinear parabolic equations with critical nonlinearities, *J. Math. Anal. Appl.* 310, 557-578 (2005).

[8] A. N. Carvalho, J. W. Cholewa, Strongly damped wave equations in $W_0^{1,p}(\Omega) \times L^p(\Omega)$, *Discrete and Continuous Dynamical Systems*, Supplement, 230-239 (2007).

[9] J. W. Cholewa, T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge, 2000.

[10] J. W. Cholewa, A. Rodriguez-Bernal, Extremal equilibria for dissipative parabolic equations in locally uniform spaces, *Math. Models Methods Appl. Sci.* 19, 1995-2037 (2009).

[11] J. W. Cholewa, A. Rodriguez-Bernal, Extremal equilibria for monotone semigroups in ordered spaces with application to evolutionary equations, *J. Differential Equations* 249, 485-525 (2010).

[12] J. W. Cholewa, A. Rodriguez-Bernal, "Linear and semilinear higher order parabolic equations in $\mathbb{R}^N$", Serie de Prepublicaciones del Departamento de Matematica Aplicada, U. Complutense, MA-UCM 2011-12, http://www.mat.ucm.es/deptos/ma.

[13] T. Dlotko, C. Sun, Dynamics of the modified viscous Cahn-Hilliard equation in R^N, *Topol. Methods Nonlinear Anal.*, 35, 2010, 277-294.

[14] M. A. Efendiev, S. V. Zelik, The attractor for a nonlinear reaction-diffusion system in an unbounded domain, *Comm. Pure Appl. Math.* 54, 625-688 (2001).

[15] E. Feireisl, Ph. Laurencot, F. Simondon, H. Toure, Compact attractors for reaction-diffusion equations in R^n, C. R. Acad. Sci. Paris Ser. I 319, 147-151 (1994).

[16] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence, R. I., 1988.

[17] D. Henry, *Geometric Theory of Semilinear Parabolic Equations,* Lecture Notes in Mathematics 840, Springer, Berlin, 1981.

[18] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Rat. Mech. Anal.* 58, 181-205 (1975).

[19] S. Merino, On the existence of the compact global attractor for semilinear reaction diffusion systems on R^N, *J. Differential Equations* 132, 87-106 (1996).

[20] A. Rodriguez-Bernal, A. Vidal-López, Semistable extremal ground states for nonlinear evolution equations in unbounded domains, *J. Math. Anal. Appl.* 338, 675-694 (2008).

[21] A. Rodriguez-Bernal, B. Wang, Attractors for partly dissipative reaction diffusion systems in R^N, *J. Math. Anal. Appl.* 252, 790-803 (2000).

[22] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Veb Deutscher, Berlin, 1978.

[23] B. Wang, Attractors for reaction di!usion equations in unbounded domains, *Physica D* 128, 41–52 (1999).

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