Asymptotic behavior of reaction diffusion equations in weighted Sobolev spaces *

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1 Introduction

In this paper we study the asymptotic behavior of solutions of reaction diffusion equations of the form

$$\begin{cases} u_t - \Delta u = f(x,u), & \text{for } x \in \mathbb{R}^N,\ t>0 \\ u(0) = u_0 \end{cases} \tag{1.1}$$

where the initial value and the solutions are in suitable weighted Sobolev spaces.

The fact that equation (1.1) is posed in the whole $\mathbb{R}^N$ introduces some difficulties related to the unbounded character of the domain. If for instance, we try to study this equation in standard Lebesgue spaces $L^p(\mathbb{R}^N)$ we have to cope with the problem that $L^p(\mathbb{R}^N)$ spaces are not nested. In particular, constant functions are not contained in $L^p(\mathbb{R}^N)$ for $1 \le p < \infty$ and therefore, if, for instance, the nonlinearity is of the form $f(x,u) = f(u)$, the roots of f cannot be considered as equilibria of the equation since they do not live in the space. Also, traveling wave solutions connecting different roots of $f(u)$ present the same difficulty. Note that working in $L^\infty(\mathbb{R}^N)$ does not help, since for example, traveling waves do not connect different roots of $f(u)$ in such a norm.

Another difficulty is that the standard Sobolev embeddings are not compact. This lack of compactness does not allow to deduce compactness properties of the semigroup, which is a clear drawback for the proof of the existence of attractors.

*Partially suported by Project MTM2006–08262, DGES, Spain, GR58/08, Grupo 920894 (BSCH-UCM, Spain). Moreover, the first and third authors are also supported by PHB2006-003 PC from MICINN and the first author is also supported by SIMUMAT, Comunidad de Madrid

Hence, standard nonlinearities like $f(u) = u - u^3$ which have extensevely studied and are nowadays well understood if the domain were bounded, present serious difficulties when one is faced with the problem of the asymptotic behavior of solutions in $\mathbb{R}^N$.

These facts indicate that we should look for other appropriate functional settings to study equation (1.1). Nevertheless, even in the setting of standard Lebesgue spaces there are some interesting studies of the dynamics of the equation above, see for instante [7] and references therein. Note also that [7], [26] deal with the case of unweighted spaces, while [6, 8, 13] work in the so called locally uniform spaces, which are intermediate between unweighted and weighted spaces. A setting in BUC can be found n [21, 16].

Observe that the problem for $f(u) = u - u^3$ can not be handled within the results in [7], [26] nor [21, 16], although it is dissipative, in some sense, in locally uniform spaces, see [8, 13].

With regards to the use of weighted Sobolev spaces, one of the pioneer works is [11], where they propose the analysis of a problem like (1.1) in weighted spaces. The nonlinearity they consider is of the type $f(x, u) = \lambda_0 u + f_0(u) + g(x)$ where $\lambda_0 < 0$ and f_0 has some strong growth conditions and several restrictive sign conditions. They analyze the equation in $L^2_\rho(\mathbb{R}^N)$ with $\rho(x) = (1 + |x|)^\gamma$. Where $L^p_\rho(\mathbb{R}^N)$ is is defined as the set of functions $u \in L^1_{loc}(\mathbb{R}^N)$ such that $u\rho^{1/p} \in L^p(\mathbb{R}^N)$ if $1 \leq p < \infty$ and $u\rho \in L^\infty(\mathbb{R}^N)$ if $p = +\infty$. In particular, if $\gamma < -N/2$ the constant functions are in $L^p_\rho(\mathbb{R}^N)$ for all $1 \leq p < \infty$. They obtain attractors in some weak topology in the case $\gamma < 0$, and in strong ones if $\gamma > 0$. Furhter developments of this theory are obtained in [18, 30, 24]. We would also like to mention the works [1] and [17] where they consider nonlinearities depending also on ∇u.

The accomplishments of the articles mentioned above are very valuable and worth to be mentioned although from a detailed study of their results it seems clear that a satisfactory linear and nonlinear theory for this type of equations in weighted Sobolev spaces neededs to be completed. With respect to linear problems such a theory must include several important aspects. One of them is a deep analysis of the regularization properties of the linear heat and Schrodinger semigroups in the class of Sobolev weighted spaces, including $L^p - L^q$ type estimates, which are very well known for standard unweighted L^p spaces, see [27], and play a central role in the analysis of nonlinear equations; see also Section B6 in [27] for some results in weighted spaces. Other important aspect in the linear theory, related with the first one, is the analysis of the generation of analytic semigroups and the characterization of their fractional power spaces, see [20] for a general theory. These goals cannot be accomplished without analyzing in detail some functional properties of Sobolev spaces with weights, like sobolev embeddings, density properties and so forth. With respect to nonlinear problems, the theory should include a general theory of local and global existence of solutions, a deep analysis of dissipative mechanisms conditions that guarantee that solutions eventually enter in a bounded set of the space and, very important, conditions that imply some compactness property of the semigroup that ultimately leads to the existence of the global attractor.

As a matter of fact, having a close look at the papers mentioned above, see for instance [11, 17], we may observe that to obtain local (and global) existence of solutions for the nonlinear problems it is used in a strong way the particular structure of the nonlinearity. Actually, the fact that $f(x, u) = \lambda_0 u + f_0(u) + g(x)$, with $\lambda_0 < 0$ and $f_0(u)u \leq 0$, is used to obtain existence of solutions for the nonlinear problems. Moreover, the proof supplied in this papers does not apply for local existence of solutions for functions of the type $f(x, u) = u|u|^{p-1}$ for any $p > 1$. As a matter of fact local and global existence of solutions is obtained at once and the hypotheses $\lambda_0 < 0$ and $f_0(u)u \leq 0$ are used in a essential way. This is in sharp contrast with the standard theory of semilinear parabolic equations where only some growth condition of the nonlinearity is needed to obtain a well posed problem locally in time, see [20]. An explanation about this fact can be found in the paper [10] where it is shown that problem (1.1) with $f(x, u) = u|u|^{p-1}$ is not even well posed locally in time for any $p > 1$ in any space $L^q_\rho(\mathbb{R}^N)$, $q \geq 1$, for any weight ρ satisfying $\rho(x) \to 0$ as $|x| \to +\infty$. It is actually shown in [10] that there exist sequences of smooth, compactly supported initial conditions $u_0^n \in L^q_\rho(\mathbb{R}^N)$ with $u_0^n \to 0$ in $L^q_\rho(\mathbb{R}^N)$, and the time of existence of the solution starting at u_0^n approaches 0 as $n \to +\infty$. This fact leaves little hope to obtain a general local existence result for nonlinearities $f(x, u)$ satisfying growth estimates of the type $|\frac{\partial f}{\partial u}(x, u)| \leq C(1 + |u|^{p-1})$ for some $p > 1$ which could obtained via fixed point arguments in the variation of constants formula, see for instance [20]. Hence, to show that problem (1.1) is well posed in spaces with weights for nonlinearities of the type $f(x, u) = \lambda_0 u + f_0(u) + g(x)$ the function f_0 must satisfy some strong sign conditions.

All this problematic suggest, as we do in this paper, to consider nonlinearities that depend on the spatial variable x, that is, of the form $f(x, u)$, and to determine a class of nonlinear terms such that the problem is well posed in weighted spaces. Thus, as we will show, the spatial behavior of the nonlinear term is somehow related to the behavior of the weight. Also, in this direction, in this paper we give a suitable theory, as sketched above, in $L^q_\rho(\mathbb{R}^N)$ in the case $\lim_{|x|\to\infty} \rho(x) = 0$, and in also in the case $\lim_{|x|\to\infty} \rho(x) = \infty$.

We describe now the contents of our paper.

In Section 2 we introduce a class of weights, that may go to zero or to infinity as $|x| \to +\infty$, define the corresponding Sobolev spaces with these weights and analyze the most important properties of these spaces. In particular, we see that if the weight considered decays to zero as $|x| \to +\infty$, the Sobolev embeddings are of the type $W^{s,p}_\rho(\mathbb{R}^N) \hookrightarrow W^{r,q}_{\rho^{q/p}}(\mathbb{R}^N)$, with $q/p > 1$, and no Sobolev embedding of the type $W^{s,p}_\rho(\mathbb{R}^N) \hookrightarrow W^{r,q}_\rho(\mathbb{R}^N)$ can be obtained. On the other hand if rh weight goes to infinity as $|x| \to \infty$ then Sobolev type embeddings, similar to the one in unweighted spaces, hold.

In Section 3 we develop the linear theory for heat ($u_t - \Delta u = 0$) and Schrödinger ($u_t - \Delta u + V(x)u = 0$) linear equations in weighted Sobolev spaces. We show that they generate analytic semigroups, establish concrete weighted $L^p - L^q$ estimates for these two equations and analyze the exponential type of the semigroups. Note here that differential operators like $-\Delta$ or $-\Delta + V(x)$ are not selfadjoint in $L^2_\rho(\mathbb{R}^N)$, see [14, 15].

In Section 4 we analyze the nonlinear evolutionary problem. We give appropriate growth conditions on the nonlinearities guaranteeing a local existence theorem, see Theorem 4.1. Later on, we impose some conditions on the nonlinearity (which are of the type $f(x, u)u \leq C(x)u^2 + D(x)|u|$ for some appropriate functions $C(x)$, $D(x)$) that guarantee global existence of solutions, see Theorem 4.3, and that the flow generated by the nonlinear equation is dissipative, that is, that we have a bounded absorbing set.

In Section 5 we study the compactness properties of the nonlinear semigroups and show that the system has a global attractor in weighted Sobolev spaces. We also analyze other important properties of the asymptotic behavior of the flows, like the existence of the so-called extremal equilibria (see [25, 26, 13]), that is, two equilibria $\varphi_m \leq \varphi_M$ of the equation with the property that all the asymptotic dynamics of the system is contained in the "interval" $[\varphi_m, \varphi_M] = \{u(x) : \varphi_m(x) \leq u(x) \leq \varphi_M(x)\}$.

Acknowledgements. Part of this work is taken from the Ph. D. Thesis of the second author, see [22], done at the Departamento de Matemática Aplicada at Universidad Complutense de Madrid. This author wish to express her gratitude to the Department.

2 Weighted spaces

A weight is a continuous and strictly positive function $\rho : \mathbb{R}^N \to (0, \infty)$.

The following class of weights was used in [7].

Definition 2.1. *We say that a weight function $\rho : \mathbb{R}^N \to (0,\ \infty)$ is in the class $R_{\rho_1,\dots,\rho_k}$, with $k \in \mathbb{N}$ and $\rho_j > 0$ if:*
i) $\rho \in C^k(\mathbb{R}^N)$
ii) $|D^\alpha \rho(x)| \le \rho_j \rho(x)$, for $x \in \mathbb{R}^N$, $|\alpha| = j$ with $1 \le j \le k$.

In particular, we say that ρ is in the class R_∞ if the condition above is satisfied for every $k \in \mathbb{N}$.

Example 2.2.
i) The weight $\rho(x) := (1 + |x|^2)^\gamma$, $x \in \mathbb{R}^N$, $\gamma \in \mathbb{R}$, belongs to the class R_{ρ_1,ρ_2} with $\rho_1 = 2|\gamma|$, $\rho_2 = 4|\gamma||\gamma - 1| + 2|\gamma|$.
ii) Assume $\rho(x)$ is a $C^2(\mathbb{R}^N)$ weight such that $\rho(x) = e^{\gamma|x|}$, $\gamma \in \mathbb{R}$ for all $|x| \ge 1$ then $\rho(x)$ is in the class R_{ρ_1,ρ_2}, for some ρ_1, ρ_2 that depend continuously on γ.

Now we summarize some properties of this class of weights that will be used further below.

Lemma 2.3. *Assume ρ is in the class R_{ρ_1,ρ_2}. Then*
(i) The weight $\rho_\varepsilon(x) := \rho(\varepsilon x)$ is in the class $R_{\varepsilon\rho_1,\varepsilon^2\rho_2}$.
(ii) We have $\rho(x) \le \rho(x - y)e^{\sqrt{N}\rho_1|y|}$, $x,\ y \in \mathbb{R}^N$. In particular $\rho(x) \le e^{C|x|}\rho(0)$, where $C = \sqrt{N}\rho_1$. That is, the weights of the class R_{ρ_1,ρ_2} have at most an exponential growth in infinity and moreover

$$\frac{\rho(x)}{\rho(y)} \le e^{\sqrt{N}\rho_1|x-y|}.$$

(iii) If $\rho(x) = (1 + |x|^2)^\gamma$, with $\gamma \in \mathbb{R}$, we have

$$\frac{\rho(x)}{\rho(y)} \le C(\gamma)(1 + |x - y|^2)^{|\gamma|}$$

and $\rho \in L^1(\mathbb{R}^N)$ iff $\gamma < \frac{-N}{2}$.
(iv) If $\rho \in C^2(\mathbb{R}^N)$, is such that $\rho(x) = e^{\gamma|x|}$ for $|x| \ge 1$, with $\gamma \in \mathbb{R}$, then

$$\frac{\rho(x)}{\rho(y)} \le Ce^{|\gamma||x-y|} \text{ for each } x, y \in \mathbb{R}^N$$

and $\rho \in L^1(\mathbb{R}^N)$ if $\gamma < 0$.

Proof. Parts (i) and (ii) are immediate.

(iii) This is equivalent to prove that $\rho(x)\rho(z) \le C\rho(x + z)$, for each $x, z \in \mathbb{R}^N$, if $\gamma < 0$ and $\rho(x)\rho(z) \ge C\rho(x + z)$, for each $x, z \in \mathbb{R}^N$, if $\gamma > 0$.

From the Cauchy-Schwartz inequality

$$1+|x+z|^2 \le 2(1+|x|^2)(1+|z|^2).$$

From here, if $\gamma<0$ we have $[(1+|x|^2)(1+|z|^2)]^\gamma \le (\frac{1}{2})^\gamma(1+|x+z|^2)^\gamma$ and if $\gamma>0$ then we have $(1+|x+z|^2)^\gamma \le 2^\gamma[(1+|x|^2)(1+|z|^2)]^\gamma$. In both cases we get the result. On the other hand, the integrability condition is obvious.

(*iv*) For $x\in B(0,1)$ and some $m_0=m_0(\gamma)$ we have

$$\rho(x) \le \max_{z\in B(0,1)} \rho(z) := M_0 \le m_0 \min_{z\in B(0,1)} e^{\gamma|z|} \le m_0 e^{\gamma|x|}. \tag{2.2}$$

Also, for each $y\in B(0,1)$ and some $m_1=m_1(\gamma)$, we have

$$e^{\gamma|y|} \le \max_{z\in B(0,1)} e^{\gamma|z|} \le m_1 \min_{z\in B(0,1)} \rho(z) \le m_1\rho(y). \tag{2.3}$$

From (2.2) and (2.3) we get

$$\frac{\rho(x)}{\rho(y)} \le \frac{m_0m_1e^{\gamma|x|}}{e^{\gamma|y|}} = m_0m_1e^{\gamma[|x|-|y|]}, \text{ for each } x,y\in B(0,1), \text{ with } \gamma\in\mathbb{R}. \tag{2.4}$$

On the other hand, for $|x|>1$, $|y|>1$

$$\frac{\rho(x)}{\rho(y)} = \frac{e^{\gamma|x|}}{e^{\gamma|y|}} = e^{\gamma[|x|-|y|]}. \tag{2.5}$$

When $|x|>1$ and $y\in B(0,1)$, we use (2.3) and the definition of $\rho(x)$ to obtain

$$\frac{\rho(x)}{\rho(y)} \le \frac{e^{\gamma|x|}m_1}{e^{\gamma|y|}} = m_1e^{\gamma[|x|-|y|]}. \tag{2.6}$$

Finally, if $x\in B(0,1)$ and $|y|>1$, using (2.2) and the definition of $\rho(x)$

$$\frac{\rho(x)}{\rho(y)} \le \frac{e^{\gamma|x|}m_0}{e^{\gamma|y|}} = m_0e^{\gamma[|x|-|y|]}. \tag{2.7}$$

From (2.4), (2.5), (2.6) and (2.7) we take $C=\max\{m_0m_1,1,m_1,m_0\}$ to obtain,

$$\frac{\rho(x)}{\rho(y)} \le C(\gamma)e^{\gamma[|x|-|y|]} \le C(\gamma)e^{\gamma|x-y|}, \text{ for all } x,y\in\mathbb{R}^N, \text{ with } \gamma\in\mathbb{R}. \square$$

Now we have the following lemma which will be useful in the next section.

Lemma 2.4. *If* $\rho\in R=R_{\rho_1,\rho_2,\ldots,\rho_n}$ *then*

$$|D^\alpha\rho^w(x)| \le C\rho^w(x), \text{ for all } |\alpha|\le n, \text{ and } w\in\mathbb{R}.$$

where $C=C(n,w,\rho_1,\rho_2,....,\rho_n)$.

In particular, for every $w\in\mathbb{R}$, $\rho^w\in R_{\hat\rho_1,\ldots,\hat\rho_n}$ *with* $\hat\rho_1=\ldots=\hat\rho_n=C$, *and if* $w\in(0,1)$ *then* $\hat\rho_i$ *can be taken independent of* w.

Proof. We proceed by induction in the order of α.

1) (a) We prove the result for $|\alpha| = 1$. taking derivatives and using the assumptions we have

$$|\partial_i \rho^w(x)| = |w| \rho^{w-1}(x) |\partial_i \rho(x)| \leq \rho_1 |w| \rho^w(x). \tag{2.8}$$

(b) If $|\alpha| = 2$.

$$\begin{aligned} |\partial_j \partial_i \rho^w(x)| &= |\ \partial_j [w \rho^{w-1}(x) \partial_i \rho(x)]\ | = |\ w[\rho^{w-1}(x) \partial_j \partial_i \rho(x) + \partial_i \rho(x) \partial_j \big(\rho^{w-1}(x)\big)]\ | \\ &\leq \Big(|w| \rho_2 + |w||w-1| \rho_1^2\Big) \rho^w(x). \end{aligned}$$

where we have used (2.8).

2) Assume the result for $|\alpha| \leq n$ and for all $w \in \mathbb{R}$. Then we show it holds for $|\alpha| = n+1$ and $w \in \mathbb{R}$. From the case $|\alpha| = 1$, using Leibniz's rule, the induction assumption and $\rho \in R = R_{\rho_1, \rho_2, \ldots, \rho_{n+1}}$ we have, for some $\hat{\alpha}$ with $|\hat{\alpha}| = n-1$,

$$\begin{aligned} D^\alpha \rho^w(x) &= (D^{\hat{\alpha}} \partial_i) \rho^w(x) = D^{\hat{\alpha}} [w \rho^{w-1}(x) \partial_i \rho(x)] \leq |w \sum_{\beta \leq \hat{\alpha}} C_{\hat{\alpha}, \beta} D^\beta (\rho^{w-1}(x)) D^{\hat{\alpha} - \beta} (\partial_i \rho(x))| \\ &\leq \sum_{\beta \leq \hat{\alpha}} \hat{C}_{\hat{\alpha}, \beta} \rho^{w-1}(x) |D^{\hat{\alpha} - \beta + 1} \rho(x)|. \end{aligned}$$

Hence $|D^\alpha \rho^w(x)| \leq \sum_{\beta \leq \hat{\alpha}} \hat{C}_{\hat{\alpha}, \beta} \rho^{w-1}(x) C_{\hat{\alpha} - \beta + 1} \rho(x) = C_1 \rho^w(x)$.□

Given a weight $\rho(x)$, we define the weighted Sobolev spaces as follows, see also [6].

Definition 2.5.
i) For $1 \leq p < \infty$, we define

$$L^p_\rho(\mathbb{R}^N) := \{u \in L^p_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^p \rho(x)\, dx < \infty\},\ 1 \leq p < \infty$$

with norm $\|u\|_{L^p_\rho(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |u(x)|^p \rho(x)\, dx \Big)^{\frac{1}{p}}$.
ii) For $p = \infty$ we define

$$L^\infty_\rho(\mathbb{R}^N) := \{u \in L^\infty_{loc}(\mathbb{R}^N) : \sup_{x \in \mathbb{R}^N} |u(x)| \rho(x) < \infty\}$$

with norm $\|u\|_{L^\infty_\rho(\mathbb{R}^N)} := \sup_{x \in \mathbb{R}^N} |u(x)| \rho(x)$.

In a similar way, we define the weighted Sobolev spaces $W^{k,p}_\rho(\mathbb{R}^N)$.

Definition 2.6. *For $k \in \mathbb{N}$, $1 \leq p \leq \infty$ we denote $W^{k,p}_{loc}(\mathbb{R}^N)$ the space of $\phi \in L^p_{loc}(\mathbb{R}^N)$ with distributional derivatives $D^\alpha \phi \in L^p_{loc}(\mathbb{R}^N)$ for all $|\alpha| \leq k$.*

We also define $W^{k,p}_\rho(\mathbb{R}^N)$ as the Banach space of all $\phi \in W^{k,p}_{loc}(\mathbb{R}^N)$ such that the norm

$$\|\phi\|_{W^{k,p}_\rho(\mathbb{R}^N)} := \sum_{|\alpha| \leq k} \|D^\alpha \phi\|_{L^p_\rho(\mathbb{R}^N)} < \infty.$$

Remark 2.7.
i) If the weight satisfies $0 < m \leq \rho(x) \leq M$*, for* $x \in \mathbb{R}^N$*, then the spaces* $L^p_\rho(\mathbb{R}^N)$ *coincide with the spaces* $L^p(\mathbb{R}^N)$*, with equivalent norms. In the same way the weighted Sobolev spaces coincide with the standard Sobolev spaces* $W^{k,p}(\mathbb{R}^N)$*.*

Therefore, the natural case to consider are: a) weights that go to infinity at infinity and b) weights that go to zero at infinity.
ii) Observe that if the weight verifies that for all small $\varepsilon > 0$*,*

$$\lim_{|x|\to\infty} \frac{\rho(\varepsilon x)}{\rho(x)} = C_\varepsilon > 0$$

then the weights $\rho(x)$ *and* $\rho_\varepsilon(x) = \rho(\varepsilon x)$ *define the same spaces, with equivalent norms.*

This property is satisfied by the weights $\rho(x) := (1+|x|^2)^\gamma$*,* $\gamma \in \mathbb{R}$*, but not the* $C^2(\mathbb{R}^N)$ *weight such that* $\rho(x) = e^{\gamma|x|}$*, for all* $|x| \geq 1$*,* $\gamma \in \mathbb{R}$*.*

Now we present some relationships between weighted and unweighted Sobolev spaces.

Lemma 2.8. (i) *If* $\rho \in L^1(\mathbb{R}^N)$ *then* $L^\infty(\mathbb{R}^N) \hookrightarrow L^p_\rho(\mathbb{R}^N)$*, for each* $1 \leq p < \infty$ *and if* $p \geq q$ *then* $L^p_\rho(\mathbb{R}^N) \hookrightarrow L^q_\rho(\mathbb{R}^N)$*.*

(ii) *In any case* $L^p(\mathbb{R}^N) \cap L^p_\rho(\mathbb{R}^N)$ *is dense in* $L^p_\rho(\mathbb{R}^N)$*, for* $1 \leq p < \infty$*.*

Proof. (i) The first part is immediate. If $p > q \geq 1$, then we have for a suitable $0 < \alpha < 1$, to be chosen:

$$\int_{\mathbb{R}^N} |u(x)|^q \rho(x)\, dx \leq \quad \left[\int_{\mathbb{R}^N} |u(x)|^p \rho(x)^{\frac{\alpha p}{q}}\, dx\right]^{\frac{q}{p}} \left[\int_{\mathbb{R}^N} \rho(x)^{\frac{(1-\alpha)p}{p-q}}\, dx\right]^{\frac{p-q}{p}}$$

Taking $\frac{\alpha p}{q} = 1$ and from the integrability of $\rho(x)$ we get that

$$\|u\|_{L^q_\rho(\mathbb{R}^N)} \leq \|u\|_{L^p_\rho(\mathbb{R}^N)} (\int_{\mathbb{R}^N} \rho(x)\, dx)^{\frac{1}{q}}.$$

(ii) It is easy to see that given $f \in L^p_\rho(\mathbb{R}^N)$ the sequence $\{f_n\} \subset L^p(\mathbb{R}^N) \cap L^p_\rho(\mathbb{R}^N)$,

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in B(0,n) \\ 0, & \text{if, } x \notin B(0,n). \end{cases}$$

satisfies $f_n \to f$ in $L^p_\rho(\mathbb{R}^N)$.$\square$

Now we establish an isometric isomorphism between weighted and unweighted Sobolev spaces which will be used to prove several properties of the former ones.

Proposition 2.9. *Assume* $1 \leq p < \infty$*,* $k \in \mathbb{N} \cup \{0\}$ *and* $\rho \in R_{\rho_1,\ldots,\rho_k}$*. The mapping*

$$J_p : W^{k,p}_\rho(\mathbb{R}^N) \to W^{k,p}(\mathbb{R}^N), \qquad J_p(u) := u\rho^{1/p}$$

is an isomorphism which is moreover an isometry if $k = 0$*.*

In case $p = \infty$*, the mapping*

$$J_\infty : W^{k,\infty}_\rho(\mathbb{R}^N) \to W^{k,\infty}(\mathbb{R}^N), \qquad J_\infty(u) := u\rho$$

is an isomorphism which is moreover an isometry if $k = 0$*.*

Proof. Let $u \in W^{k,p}_\rho(\mathbb{R}^N)$ then we show that $u\rho^{\frac{1}{p}} \in W^{k,p}(\mathbb{R}^N)$. By Leibniz's rule and Lemma 2.4 we have,

$$\begin{aligned}
\|J_p(u)\|^p_{W^{k,p}(\mathbb{R}^N)} : &= \sum_{|\alpha|\le k} \|D^\alpha(u\rho^{\frac{1}{p}})\|^p_{L^p(\mathbb{R}^N)} \\
&\le \sum_{|\alpha|\le k}\sum_{\beta\le\alpha} \int_{\mathbb{R}^N} \hat{C}_{\alpha,\beta}|D^\beta u(x)|^p|D^{\alpha-\beta}\rho^{\frac{1}{p}}(x)|^p\,dx \\
&\le \sum_{|\beta|\le k} \hat{C}_\beta \int_{\mathbb{R}^N} |D^\beta u(x)|^p\rho(x)\,dx.
\end{aligned}$$

From here we get $\|J_p(u)\|^p_{W^{k,p}(\mathbb{R}^N)} \le C_1\|u\|^p_{W^{k,p}_\rho(\mathbb{R}^N)}$. Consequently, the mapping J_p is well defined, one to one and continuous. By the open mapping theorem, the inverse is continuous. Hence, it is an isomorphism, If $k = 0$ we have

$$\|u\|^p_{L^p_\rho(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u|^p\rho\,dx = \|u\rho^{\frac{1}{p}}\|^p_{L^p(\mathbb{R}^N)} = \|J_p(u)\|^p_{L^p(\mathbb{R}^N)}$$

that is, J_p is an isometry.

The case $p = \infty$ is analogous.□

The next definition holds for any interpolation method. For details see [6]. We will use below the complex interpolation method, because in this case we can characterize the fractional power spaces, as we will show below. See [29].

Definition 2.10. *For $1 \le p \le \infty$, $k \in \mathbb{N} \cup \{0\}$ and $s \in (k, k+1)$ we define $\theta \in (0,1)$ such that $s = (1-\theta)k + \theta(k+1)$, that is $\theta = s - k$. Then we define the intermediate spaces as*

$$W^{s,p}(\mathbb{R}^N) := [W^{k+1,p}(\mathbb{R}^N), W^{k,p}(\mathbb{R}^N)]_\theta,$$

and

$$W^{s,p}_\rho(\mathbb{R}^N) := [W^{k+1,p}_\rho(\mathbb{R}^N), W^{k,p}_\rho(\mathbb{R}^N)]_\theta.$$

where $[\cdot,\cdot]_\theta$ is the complex interpolation funtor, see [2]

From Proposition 2.9 and the properties of interpolation we get then

Lemma 2.11. *Let $1 \le p < \infty$, $k \in \mathbb{N}$, $\rho \in R_{\rho_1,\rho_2,....,\rho_k}$, and $0 \le s \le k$ then*

$$J_p : W^{s,p}_\rho(\mathbb{R}^N) \to W^{s,p}(\mathbb{R}^N), \qquad J_p(u) = u\rho^{1/p}$$

is an isomorphism.

If $p = \infty$, then

$$J_\infty : W^{s,\infty}_\rho(\mathbb{R}^N) \to W^{s,\infty}(\mathbb{R}^N), \qquad J_\infty(u) = u\rho$$

is an isomorphism.□

Now we establish Sobolev type inclusions between the spaces $W^{k,p}_\rho(\mathbb{R}^N)$ and $L^q_{\rho^{\frac{q}{p}}}(\mathbb{R}^N)$ with weights in the class $R_{\rho_1,\rho_2,...,\rho_k}$.

Lemma 2.12.
i) Let $1 < p < \infty$, $k \in \mathbb{N}$, $\rho \in R_{\rho_1,\ldots,\rho_k}$ *and* $0 \le s \le k$. *Then*

$$W^{s,p}_{\rho}(\mathbb{R}^N) \hookrightarrow \begin{cases} L^q_{\rho^{\frac{q}{p}}}(\mathbb{R}^N) \text{ with } p \le q \begin{cases} \le \frac{Np}{N-sp}, & \text{if } s < \frac{N}{p} \\ < \infty, & \text{if } s \ge \frac{N}{p} \end{cases} \\ L^{\infty}_{\rho^{\frac{1}{p}}}(\mathbb{R}^N), \text{ if } s > \frac{N}{p}. \end{cases}$$

ii) Let $1 < p \le q < \infty$, $k \in \mathbb{N}$, $\rho \in R_{\rho_1,\ldots,\rho_k}$, $0 \le \sigma \le s \le k$, *and* $s - \frac{N}{p} \ge \sigma - \frac{N}{q}$. *Then*

$$W^{s,p}_{\rho}(\mathbb{R}^N) \subset W^{\sigma,q}_{\rho^{\frac{q}{p}}}(\mathbb{R}^N).$$

If $q = \infty$, *under the same conditions above we have the embedding*

$$W^{s,p}_{\rho}(\mathbb{R}^N) \subset W^{\sigma,\infty}_{\rho^{\frac{1}{p}}}(\mathbb{R}^N).$$

Proof. Case i) is a particular case of case ii) with $\sigma = 0$. For case ii), if $q < \infty$, consider the following commutative diagram

$$\begin{array}{ccc} W^{s,p}_{\rho}(\mathbb{R}^N) & \stackrel{i}{\hookrightarrow} & W^{\sigma,q}_{\rho^{\frac{q}{p}}}(\mathbb{R}^N) \\ \downarrow J_p & & \uparrow J_q^{-1} \\ W^{s,p}(\mathbb{R}^N) & \stackrel{i}{\hookrightarrow} & W^{\sigma,q}(\mathbb{R}^N) \end{array}$$

Let $u \in W^{s,p}_{\rho}(\mathbb{R}^N)$ then by Lemma 2.11 we have that $J_p(u) := u\rho^{\frac{1}{p}} \in W^{s,p}(\mathbb{R}^N)$. From hypothesis $s - \frac{N}{p} \ge \sigma - \frac{N}{q}$ and then

$$W^{s,p}(\mathbb{R}^N) \hookrightarrow W^{\sigma,q}(\mathbb{R}^N).$$

Then $u\rho^{\frac{1}{p}} \in W^{\sigma,q}(\mathbb{R}^N)$, and now we determine the weight φ such that using the isomorphism in Lemma 2.11,

$$J_q : W^{\sigma,q}_{\varphi}(\mathbb{R}^N) \to W^{\sigma,q}(\mathbb{R}^N), \qquad J_q(u) = u\varphi^{1/q}$$

we have $u\varphi^{1/q} = u\rho^{\frac{1}{p}}$. From here $\varphi = \rho^{\frac{q}{p}}$ and thus we get, $u \in W^{\sigma,q}_{\rho^{\frac{q}{p}}}(\mathbb{R}^N)$.

The case $q = \infty$ is analogous. □

We end this section with some remarks about the existence of Sobolev like embeddings for weighted spaces. The existence of such embeddings will be of importance for the evolution problems.

Remark 2.13. 1.– *For weights* $\rho(x) \to \infty$, *as* $|x| \to \infty$ *and, say,* $\rho(x) \ge 1$, *for example,* $\rho(x) = (1 + |x|^2)^{\gamma}$ *with* $\gamma > 0$, *if* $q > p$ *we have the following inclusion*

$$L^q_{\rho^{\frac{q}{p}}}(\mathbb{R}^N) \hookrightarrow L^q_{\rho}(\mathbb{R}^N).$$

In fact, if $u \in L^q_{\rho^{\frac{q}{p}}}(\mathbb{R}^N)$ *then since* $\frac{q}{p} > 1$*, we have*

$$\int_{\mathbb{R}^N} |u(x)|^q \rho(x)\, dx \le \int_{\mathbb{R}^N} |u(x)|^q \rho^{\frac{q}{p}}(x)\, dx < \infty,$$

and then $u \in L^q_\rho(\mathbb{R}^N)$*.*

Hence, we have the following embeddings similar to the ones in unweighted spaces

$$W^{s,p}_\rho(\mathbb{R}^N) \hookrightarrow \begin{cases} L^q_\rho(\mathbb{R}^N) \text{ with } p \le q \begin{cases} \le q^* = \frac{Np}{N-sp}, \ if \ s < \frac{N}{p} \\ < \infty, \ if \ s \ge \frac{N}{p} \end{cases} \\ L^\infty_{\rho^{\frac{1}{p}}}(\mathbb{R}^N), \ if \ s > \frac{N}{p} \end{cases}$$

2.- *On the other hand, for weights* $\rho(x) \to 0$ *as* $|x| \to \infty$ *one can construct examples for which, for* $q > p$*,*

$$L^q_{\rho^{\frac{q}{p}}}(\mathbb{R}^N) \not\hookrightarrow L^q_\rho(\mathbb{R}^N).$$

For this, let $\rho(x) = (1+|x|^2)^\gamma$ *with* $\gamma < 0$*,* $q > p$*, and* $u(x) = (1+|x|^2)^r$*, with* r *to be chosen below. Then* $u \in L^q_{\rho^{\frac{q}{p}}}(\mathbb{R}^N)$ *if and only if* $qr + \frac{\gamma\, q}{p} < -\frac{N}{2}$*, by Lemma 2.3, which is equivalent to* $r < (\frac{-N}{2} - \frac{\gamma\, q}{p})\frac{1}{q}$*.*

On the other hand $u \notin L^q_\rho(\mathbb{R}^N)$ *if* $rq + \gamma \ge \frac{-N}{2}$*, that is* $r \ge (\frac{-N}{2} - \gamma)\frac{1}{q}$*.*

Since $q > p$ *then we can find* r *such that*

$$(\frac{-N}{2} - \gamma)\frac{1}{q} \le r < (\frac{-N}{2} - \frac{\gamma q}{p})\frac{1}{q}$$

and we get the statement.

Finally, one can easily see that $u \in W^{k,p}_\rho(\mathbb{R}^N)$ *if* $(r - \frac{k}{2})p + \gamma < \frac{-N}{2}$*, that is,* $r < (\frac{-N}{2} - \gamma)\frac{1}{p} + \frac{k}{2}$*.*

Hence, if $\gamma \le -\frac{N}{2}$*,* $q > p$ *and* $k \ge 0$*, we can chose* r *such that*

$$(\frac{-N}{2} - \gamma)\frac{1}{q} \le r < (\frac{-N}{2} - \gamma)\frac{1}{p} + \frac{k}{2}$$

and then

$$W^{k,p}_\rho(\mathbb{R}^N) \not\hookrightarrow L^q_\rho(\mathbb{R}^N)$$

and there are no inclusions as for the unweighted case.

Finally, the above holds if $-N/2 < \gamma < 0$*,* $q > p$ *and some* $k \ge 0$*.*

3 The heat and Schrödinger equations in weighted spaces.

In this section we present some results for the linear heat equation in weighted spaces $L^q_\rho(\mathbb{R}^N)$.

$$u_t - \Delta u = 0, \qquad x \in \mathbb{R}^N, \ t > 0.$$

We will prove that the realization of the linear elliptic operator $-\Delta$ in $L^q_\rho(\mathbb{R}^N)$, with a weight in the class $\rho \in R = R_{\rho_1,\rho_2}$, generates an analytic semigroup.

In fact this result is obtained from Theorem 5.1 in [6], which holds for more general elliptic operators.

Proposition 3.1. *Assume $\rho \in R_{\rho_1,\rho_2}$. For any $1 < q < \infty$ the linear unbounded operator $-\Delta$ in $L^q_\rho(\mathbb{R}^N)$, with domain $W^{2,q}_\rho(\mathbb{R}^N)$, is such that, Δ generates an order preserving analytic semigroup $\{S(t)\}_{t\geq 0}$.*

In particular, the heat equation

$$\begin{cases} u_t - \Delta u = 0, \quad x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 \in L^q_\rho(\mathbb{R}^N) \end{cases}$$

has a unique solution $u(t) := S(t)u_0$ for $t \geq 0$ which is given by

$$u(t,x) = S(t)u_0 = (4\pi t)^{\frac{-N}{2}} \int_{\mathbb{R}^N} e^{\frac{-|x-y|^2}{4t}} u_0(y)\, dy. \tag{3.1}$$

The fractional power spaces of $-\Delta$ in $L^q_\rho(\mathbb{R}^N)$, denoted $H^{\alpha,q}_\rho(\mathbb{R}^N)$ coincide with the spaces $W^{\alpha,q}_\rho(\mathbb{R}^N)$, for $0 \leq \alpha \leq 1$, given in Definition 2.10.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} L^q_\rho(\mathbb{R}^N) & \xrightarrow{-\Delta} & L^q_\rho(\mathbb{R}^N) \\ J_q \downarrow & & \uparrow J_q^{-1} \\ L^q(\mathbb{R}^N) & \xrightarrow{\Lambda} & L^q(\mathbb{R}^N). \end{array}$$

Multiplying $-\Delta u$ by $\rho^{\frac{1}{q}}$, and using $u = w\rho^{\frac{-1}{q}}$ we get

$$\Lambda w := (-\Delta u)\rho^{\frac{1}{q}} = -\Delta(w\rho^{\frac{-1}{q}})\rho^{\frac{1}{q}} = -\Delta w + \frac{2}{q}(\frac{\nabla\rho}{\rho})\nabla w + [\frac{1}{q}(\frac{\Delta\rho}{\rho}) + \frac{(-\frac{1}{q}-1)}{q})\frac{|\nabla\rho|^2}{\rho^2}]w \tag{3.2}$$

Since $\rho \in R_{\rho_1,\rho_2}$ then Λ has bounded coefficients and we can use the results in [3], to obtain that the realization of Λ in $L^q(\mathbb{R}^N)$, with domain $W^{2,q}(\mathbb{R}^N)$ is a sectorial operator in $L^q(\mathbb{R}^N)$ and the fractional power spaces coincide with the complex interpolation spaces between $W^{2,q}(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$, as in Definition 2.10. The rest follows from Theorem 5.1 [6].

That (3.1) is satisfied follows from the density in Lemma 2.8 and the estimates in Proposition 3.2, below, with $r = q$. From this, we get that the semigroup $\{S(t)\}_{t\geq 0}$ is order preserving in $L^q_\rho(\mathbb{R}^N)$. □

We preset now some results on the solution of the heat equation in the spaces $L^q_\rho(\mathbb{R}^N)$. Observe that the norm of the solution is estimated in a weighted space with a different weight than that of the initial data.

Proposition 3.2. *For each $1 \leq q \leq r < \infty$ and $u_0 \in L^q_\rho(\mathbb{R}^N)$, we have, for a certain constant M and $t > 0$*
(i) For each weight $\rho \in R_{\rho_1,\rho_2}$

$$\begin{cases} \|u(t)\|_{L^r_{\rho^{\frac{r}{q}}}(\mathbb{R}^N)} \leq Mt^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}[1 + e^{\rho_1^2 Nt}t^{\frac{N}{2}(1-\frac{1}{q}+\frac{1}{r})}]\|u_0\|_{L^q_\rho(\mathbb{R}^N)} \\ \|u(t)\|_{L^\infty_{\rho^{\frac{1}{q}}}(\mathbb{R}^N)} \leq Mt^{\frac{-N}{2q}}[1 + e^{\rho_1^2 Nt}t^{\frac{N}{2}(1-\frac{1}{q})}]\, \|u_0\|_{L^q_\rho(\mathbb{R}^N)}, \end{cases} \tag{3.3}$$

(ii) In particular if $\rho \in C^2(\mathbb{R}^N)$ is such that $\rho(x) = e^{\mu|x|}$, for $|x| \geq 1$, with $\mu \in \mathbb{R}$, we have (3.3) replacing $\rho_1^2 N$ by $|\mu|^2$,
(iii) If $\rho(x) = (1+|x|^2)^\gamma$, $\gamma \in \mathbb{R}$ then

$$\begin{cases} \|u(t)\|_{L^r_{\rho^{\frac{r}{q}}}(\mathbb{R}^N)} \leq M t^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}[1+t^{\frac{|\gamma|}{q}}]\, \|u_0\|_{L^q_\rho(\mathbb{R}^N)}. \\ \|u(t)\|_{L^\infty_{\rho^{\frac{1}{q}}}(\mathbb{R}^N)} \leq M t^{\frac{-N}{2q}}[1+t^{\frac{|\gamma|}{q}}]\, \|u_0\|_{L^q_\rho(\mathbb{R}^N)}. \end{cases}$$

Proof. Using (3.1) and multiplying by $\rho \in R_{\rho_1,\rho_2}$

$$u(t,x,u_0)\rho(x) = (4\pi t)^{\frac{-N}{2}} \int_{\mathbb{R}^N} e^{\frac{-|x-y|^2}{4t}} u_0(y)\rho(x)\, dy = (4\pi t)^{\frac{-N}{2}} \int_{\mathbb{R}^N} e^{\frac{-|x-y|^2}{4t}} \frac{\rho(x)}{\rho(y)} u_0(y)\rho(y)\, dy.$$

From Lemma 2.3, we have $\frac{\rho(x)}{\rho(y)} \leq CR(|x-y|)$ with $R(|z|) = \begin{cases} e^{\sqrt{N}\rho_1|z|}, \text{ in case (i)} \\ e^{|\mu||z|}, \text{ in case (ii)} \\ (1+|z|^2)^{|\gamma|}, \text{ in case (iii)} \end{cases}$, and then we get

$$|u(t,x,u_0)\rho(x)| \leq C(4\pi t)^{\frac{-N}{2}} \int_{\mathbb{R}^N} e^{\frac{-|x-y|^2}{4t}} R(|x-y|)\, |u_0(y)|\rho(y)\, dy.$$

Denoting $M(t)(z) := e^{\frac{-|z|^2}{4t}} R(|z|)$ we have

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq C_1(4\pi t)^{\frac{-N}{2}} \|M(t) * |u_0|\rho\|_{L^r(\mathbb{R}^N)}.$$

Young's inequlity, see [12] page 77, implies that with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$,

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq C_1(4\pi t)^{\frac{-N}{2}} \|M(t)\|_{L^p(\mathbb{R}^N)} \|u_0\rho\ \|_{L^q(\mathbb{R}^N)}. \tag{3.4}$$

Now, we estimate the integral above. For the case (i), we have

$$I(t) = \|M(t)\|_{L^p(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} e^{(\frac{-|z|^2}{4t})p} e^{\sqrt{N}\rho_1|z|p}\, dz\right)^{\frac{1}{p}} = C(\int_0^\infty e^{(\frac{-r^2}{4t}+\sqrt{N}\rho_1 r)p} r^{N-1}\, dr)^{\frac{1}{p}}.$$

Let $y(r) := \frac{-r^2}{4t} + \sqrt{N}\rho_1 r$, which is positive in $(0, 4t\sqrt{N}\rho_1)$ and negative in $(4t\sqrt{N}\rho_1, \infty)$. In the interval $[0, 4t\sqrt{N}\rho_1]$ the maximum value is attained at the point $r = 2t\sqrt{N}\rho_1$ and the maximum value is $t\rho_1^2 N$. Since $\frac{-r^2}{4t} + \sqrt{N}\rho_1 r \leq \frac{-r^2}{8t}$, if $r > 8\sqrt{N}\rho_1 t$, we get

$$I(t) \leq C\left(\int_{r\geq 8t\sqrt{N}\rho_1} e^{\frac{-r^2 p}{8t}} r^{N-1}\, dr + \int_{0\leq r\leq 8t\sqrt{N}\rho_1} e^{(\frac{-r^2}{4t}+\sqrt{N}\rho_1 r)p} r^{N-1}\, dr\right)^{\frac{1}{p}}. \tag{3.5}$$

In the inequality above the second term can be bounded above by the area of the square of base length $8\sqrt{N}\rho_1 t$ and height the maximum value of $e^{pt\rho_1^2 N}$, that is,

$$\int_{0\leq r\leq 8t\sqrt{N}\rho_1} e^{(\frac{-r^2}{4t}+\sqrt{N}\rho_1 r)p} r^{N-1}\, dr \leq e^{pt\rho_1^2 N}(8\sqrt{N}\rho_1 t)^N \tag{3.6}$$

and from (3.5) and (3.6) we obtain

$$I(t) \leq C\Big(\int_{r\geq 8t\sqrt{N}\rho_1} e^{\frac{-r^2 p}{8t}} r^{N-1}\, dr + e^{pt\rho_1^2 N}(8t\sqrt{N}\rho_1)^N \Big)^{\frac{1}{p}}.$$

Using the change of variables $r = z\sqrt{8t}$, we get

$$I(t) \leq Ct^{\frac{N}{2p}} \Big(\int_{z\geq \sqrt{8t}\sqrt{N}\rho_1} e^{-z^2 p} z^{N-1}\, dz + t^{\frac{N}{2}} e^{\rho_1^2 Ntp} \Big)^{\frac{1}{p}} = Ct^{\frac{N}{2p}} I_0^{\frac{1}{p}}(t).$$

Thus $I_0(t) = \begin{cases} \sim\ t^{\frac{N}{2}} e^{\rho_1^2 Npt} & t >> 1 \\ \leq C, \quad t\sim 0, \end{cases}$ and then $I_0(t) \leq C(1 + t^{\frac{N}{2}} e^{\rho_1^2 Ntp})$ for all $t > 0$, which implies

$$\|M(t)\|_{L^p(\mathbb{R}^N)} = I(t) \leq Ct^{\frac{N}{2p}}(1 + t^{\frac{N}{2p}} e^{\rho_1^2 Nt}). \tag{3.7}$$

Resuming the proof, from (3.4) and (3.7) we get

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq Ct^{\frac{-N}{2}(1-\frac{1}{p})}(1 + t^{\frac{N}{2p}} e^{\rho_1^2 Nt})\|u_0\rho\|_{L^q(\mathbb{R}^N)}$$

where p, q and r satisfy $1 - \frac{1}{p} := \frac{1}{q} - \frac{1}{r}$.

Finally, we get

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq Ct^{\frac{-N}{2}(\frac{1}{q}-\frac{1}{r})}\ [1 + e^{\rho_1^2 Nt} t^{\frac{N}{2}[\frac{1}{r}-\frac{1}{q}+1]}]\ \|u_0\rho\|_{L^q(\mathbb{R}^N)}. \tag{3.8}$$

Replacing ρ by $\rho^{\frac{1}{q}}$,

$$\|u(t)\|_{L^r_{\rho^{\frac{r}{q}}}(\mathbb{R}^N)} \leq Ct^{\frac{-N}{2}(\frac{1}{q}-\frac{1}{r})}\ [1 + e^{\rho_1^2 Nt} t^{\frac{N}{2}[\frac{1}{r}-\frac{1}{q}+1]}]\ \|u_0\|_{L^q_\rho(\mathbb{R}^N)}. \tag{3.9}$$

If $r = \infty$, we consider (3.8) to get

$$\|u(t)\rho\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{\frac{-N}{2q}}(1 + e^{\rho_1^2 Nt} t^{\frac{N}{2}[1-\frac{1}{q}]})\|u_0\rho\|_{L^q(\mathbb{R}^N)}. \tag{3.10}$$

Replacing ρ by $\rho^{\frac{1}{q}}$, from (3.9) and (3.10) we get (3.3).

The case (ii) is immediate.

For the case (iii), with the change $z = w\sqrt{t}$ we get in (3.4)

$$\begin{aligned}
\|M(t)\|_{L^p(\mathbb{R}^N)} &= \Big(\int_{\mathbb{R}^N} e^{(\frac{-|z|^2}{4t})p}(1+|z|^2)^{|\gamma|p}\, dz \Big)^{\frac{1}{p}} \\
&= \Big(\int_{\mathbb{R}^N} e^{(\frac{-|w|^2}{4})p}(1+|w\sqrt{t}|^2)^{p|\gamma|} t^{\frac{N}{2}}\, dw \Big)^{\frac{1}{p}} \\
&\leq t^{\frac{N}{2p}} C\Big(\int_{\mathbb{R}^N} e^{(\frac{-|w|^2}{4})p}(1+|w\sqrt{t}|^{2p|\gamma|})\, dw \Big)^{\frac{1}{p}} \\
&\leq Ct^{\frac{N}{2p}}\Big[\int_{\mathbb{R}^N} e^{(\frac{-w^2}{4})p}\, dw + t^{p|\gamma|}\int_{\mathbb{R}^N} e^{(\frac{-w^2}{4})p}|w|^{2p|\gamma|}\, dw \Big]^{\frac{1}{p}} \\
&\leq Ct^{\frac{N}{2p}}(1+t^{|\gamma|}).
\end{aligned} \tag{3.11}$$

Hence, considering (3.11) in (3.4) we arrive at

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \quad \leq \quad C_2 t^{\frac{-N}{2}[1-\frac{1}{p}]}(1+t^{|\gamma|})\|u_0\rho\|_{L^q(\mathbb{R}^N)}.$$

Since $1-\frac{1}{p} := \frac{1}{q}-\frac{1}{r}$, we obtain

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq C_2 t^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}(1+t^{|\gamma|})\|u_0\rho\|_{L^q(\mathbb{R}^N)}. \tag{3.12}$$

Replacing ρ by $\rho^{\frac{1}{q}}$, we get

$$\|u(t)\|_{L^r_{\rho^{\frac{r}{q}}}(\mathbb{R}^N)} \leq C_2 t^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}(1+t^{\frac{|\gamma|}{q}})\|u_0\|_{L^q_\rho(\mathbb{R}^N)}.$$

If $r=\infty$, from (3.12), replacing ρ by $\rho^{\frac{1}{q}}$ we get

$$\|u(t)\|_{L^\infty_{\rho^{\frac{1}{q}}}(\mathbb{R}^N)} \leq C_2 t^{\frac{-N}{2q}}(1+t^{\frac{|\gamma|}{q}})\|u_0\|_{L^q_\rho(\mathbb{R}^N)}.\Box$$

Observe that the key point in the proof above is the Gaussian structure of the heat kernel. Hence, the results above for $-\Delta$ can be obtained for other elliptic operators with a similar bound for the corresponding parabolic kernel. Therefore, using the results in [16] we have the following corollary,

Corollary 3.3. *Assume the differential operator L is given by*

$$L(u) := -\sum_{i,j=1}^{N} \partial_i\Big(a_{i,j}(x)\partial_j u + a_i(x)u\Big) + b_i(x)\partial_i u + c_0(x)u$$

with real coefficients $a_{i,j}$, a_i, b_i, c_0 in $L^\infty(\mathbb{R}^N)$ and satisfying the ellipticity condition

$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2, \text{ for some } \alpha_0>0, \text{ and for each } \xi\in\mathbb{R}^N.$$

Then the fundamental solution of the parabolic problem $u_t+Lu=0$ en $\mathbb{R}^N$ satisfies a Gaussian bound

$$0\leq k(x,y,t,s)\leq C(t-s)^{\frac{-N}{2}}e^{w(t-s)}e^{-c\frac{|x-y|^2}{(t-s)}} \text{ for } t>s \text{ and } x,y\in\mathbb{R}^N.$$

where, C, c, w depend on the L^∞ norm of the coefficients.

Therefore, the semigroup generated by $-L$ is given by

$$u(t,x) := T_L(t)u_0 = \int_{\mathbb{R}^N} k(x,y,t,0)u_0(y)\,dy$$

and satisfies the estimates in Proposition 3.2.

For its importance in applications we discuss in this section Schrödinger operators in weighted spaces, with $\rho \in R_{\rho_1,\rho_2}$. Hence, we consider the linear parabolic equation

$$\begin{cases} u_t - \Delta u = V(x)u, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 \in L^q_\rho(\mathbb{R}^N) \end{cases}$$

and we consider a class of potentials that admit local singularities and have no prescribed behavior at infinity, that we denote $L^\sigma_U(\mathbb{R}^N)$, $1 \le \sigma \le \infty$, which is defined as

$$L^\sigma_U(\mathbb{R}^N) := \{V \in L^\sigma_{loc}(\mathbb{R}^N) : \sup_{x\in\mathbb{R}^N} \int_{B(x,1)} |V(y)|^\sigma \, dy < \infty\}$$

with norm

$$\|V\|_{L^\sigma_U(\mathbb{R}^N)} := \sup_{x\in\mathbb{R}^N} \|V\|_{L^\sigma(B(x,1))}.$$

These spaces are named uniform spaces in the literature, see for instance [6] and references therein.

The results below hold for more general operators than $-\Delta$, but we will focus in operator of the form $-\Delta - V(x)I$, where V is a potential in $L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, $\sigma \ge 1$.

Then we have

Theorem 3.4. *Let $V \in L^\sigma_U(\mathbb{R}^N)$ with $\sigma > \frac{N}{2}$, $\sigma > 1$. Then for each $1 < q < \sigma$ the operator $\Delta + V(x)I$, with domain $W^{2,q}_\rho(\mathbb{R}^N)$, generates an order preserving analytic semigroup in $L^q_\rho(\mathbb{R}^N)$, $S_V(t)$, and with the same fractional power spaces than $-\Delta$.*

The semigroup is given by the variation of constants formula

$$u(t) = e^{(\Delta+V)t}u_0 = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}V(x)u(s) \, ds.$$

Proof. From Proposition 3.1 we have that for each $1 < q < \infty$ the operator Δ with domain $W^{2,q}_\rho(\mathbb{R}^N)$ generates an order preserving analytic semigroup in $L^q_\rho(\mathbb{R}^N)$.

Denote by P the operator

$$P : L^r_\rho(\mathbb{R}^N) \to L^q_\rho(\mathbb{R}^N), \qquad P(u)(x) = V(x)u(x).$$

We show below that P is bounded if $\frac{1}{\sigma} + \frac{1}{r} = \frac{1}{q}$ and $1 \le q < \sigma$. For this, we first decompose $\mathbb{R}^N$ in cubes in the following way: for each $i \in \mathbb{Z}^N$, denote by Q_i the open cube in $\mathbb{R}^N$ centered at i, with sides of length 1 and parallel to the axes. Then $Q_i \cap Q_j = \emptyset$ for $i \ne j$ and $\mathbb{R}^N = \cup_{i\in\mathbb{Z}^N} \overline{Q_i}$.

From Hölder's inequality

$$\begin{aligned} \|Vu\|^q_{L^q_\rho(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} |V(x)|^q |u(x)|^q \rho(x) \, dx = \sum_{i\in\mathbb{Z}^N} \int_{Q_i} |V(x)|^q |u(x)|^q \rho(x) \, dx \\ &\le \sum_{i\in\mathbb{Z}^N} \|V\|^q_{L^\sigma(Q_i)} \|u\rho^{\frac{1}{q}}\|^q_{L^r(Q_i)}, \text{ with } \frac{1}{\sigma} + \frac{1}{r} = \frac{1}{q} \end{aligned} \tag{3.13}$$

which is possible since $1 \le q < \sigma$.

Sobolev inclusions imply then

$$W^{2,q}(Q_i) \hookrightarrow L^s(Q_i), \text{ with } s \text{ such that } \quad 2 - \frac{N}{q} = \frac{-N}{s}$$

or equivalently $\frac{2}{N} - \frac{1}{q} = -\frac{1}{s}$, with constants independent of i. Since $\frac{1}{q} = \frac{1}{\sigma} + \frac{1}{r}$, using $\sigma > \frac{N}{2}$ we get $q < r < s$.

Interpolating with $\frac{1}{r} = \theta\frac{1}{q} + (1-\theta)\frac{1}{s}$ we get

$$\|u\rho^{\frac{1}{q}}\|^q_{L^r(Q_i)} \le \|u\rho^{\frac{1}{q}}\|^{q(1-\theta)}_{L^s(Q_i)}\|u\rho^{\frac{1}{q}}\|^{q\theta}_{L^q(Q_i)} \le \|u\rho^{\frac{1}{q}}\|^{(1-\theta)q}_{W^{2,q}(Q_i)}\|u\rho^{\frac{1}{q}}\|^{q\theta}_{L^q(Q_i)}$$

Now Young's inequality yields

$$\|u\rho^{\frac{1}{q}}\|^q_{L^r(Q_i)} \le \varepsilon\|u\rho^{\frac{1}{q}}\|^q_{W^{2,q}(Q_i)} + C_\varepsilon\|u\rho^{\frac{1}{q}}\|^q_{L^q(Q_i)},$$

and then

$$\begin{aligned} \|Vu\|^q_{L^q_\rho(\mathbb{R}^N)} &\le \sum_{i\in\mathbb{Z}^N} \|V\|^q_{L^\sigma(Q_i)}\|u\rho^{\frac{1}{q}}\|^q_{L^r(Q_i)} \\ &\le \|V\|^q_{L^\sigma_U(\mathbb{R}^N)} \sum_{i\in\mathbb{Z}^N}\left[\varepsilon\|u\rho^{\frac{1}{q}}\|^q_{W^{2,q}(Q_i)} + C_\varepsilon\|u\rho^{\frac{1}{q}}\|^q_{L^q(Q_i)}\right] \\ &\le \|V\|^q_{L^\sigma_U(\mathbb{R}^N)}\left[\varepsilon\|u\rho^{\frac{1}{q}}\|^q_{W^{2,q}(\mathbb{R}^N)} + C_\varepsilon\|u\rho^{\frac{1}{q}}\|^q_{L^q(\mathbb{R}^N)}\right]. \end{aligned}$$

Hence, relabeling the coefficients, we get

$$\|Vu\|^q_{L^q_\rho(\mathbb{R}^N)} \le \varepsilon\|u\|^q_{W^{2,q}_\rho(\mathbb{R}^N)} + C_\varepsilon\|u\|^q_{L^q_\rho(\mathbb{R}^N)}.$$

By [20], Theorem 1.3.2, we have that $\Delta + V(x)I$ generates an analytic semigroup in $L^q_\rho(\mathbb{R}^N)$.

Now we prove that $-\Delta - V(x)I$ has the same fractional power spaces than $-\Delta$. For this, by Theorem 1.4.8 in [20] it suffices to prove that

$$\|Vu\|_{L^q_\rho(\mathbb{R}^N)} \le C\|u\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}, \text{ for some } \quad 0 \le \alpha < 1.$$

We estimate above and below (3.13) and for this we use the inclusions

$$H^{2\alpha,q}(Q_i) \hookrightarrow L^r(Q_i), \text{ for } 2\alpha - \frac{N}{q} \ge \frac{-N}{r}, \ \ 0 \le \alpha < 1$$

with constants independent of i.

Using the relationship among σ, r and q as in (3.13) we get $2\alpha \ge \frac{N}{\sigma}$, which holds for some $0 \le \alpha < 1$ because $\sigma > \frac{N}{2}$. From this we get

$$\|Vu\|^q_{L^q_\rho(\mathbb{R}^N)} \le C\|V\|^q_{L^\sigma_U(\mathbb{R}^N)} \sum_{i\in Z^N}\|u\rho^{\frac{1}{q}}\|^q_{H^{2\alpha,q}(Q_i)} \le C\|V\|^q_{L^\sigma_U(\mathbb{R}^N)}\|u\rho^{\frac{1}{q}}\|^q_{H^{2\alpha,q}(\mathbb{R}^N)}$$

where the last inequality comes from Lemma 2.4 in [7]. Since the norms $\|u\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}$ and $\|u\rho^{\frac{1}{q}}\|_{H^{2\alpha,q}(\mathbb{R}^N)}$, are equivalent, we get

$$\|Vu\|_{L^q_\rho(\mathbb{R}^N)} \le C\|V\|_{L^\sigma_U(\mathbb{R}^N)}\|u\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}, \text{ for some } 0 < \alpha < 1.$$

To prove the order preserving property observe that from the results in [27], the semigroup generated by $\Delta + V(x)I$ is order preserving in $L^q(\mathbb{R}^N)$, for $1 \le q \le \infty$. Hence, from density, see Lemma 2.8, we get the result.□

In the following result we prove weighted $L^q - L^r$ estimates for the Schrödinger semigroup, analogous to the ones for the heat equation.

Theorem 3.5. *Let $\rho \in R_{\rho_1,\rho_2}$, $V \in L^\sigma_U(\mathbb{R}^N)$ with $\sigma > \frac{N}{2}$. Then there exists constants a and M depending only on N, σ and $\|V\|_{L^\sigma_U(\mathbb{R}^N)}$ such that for each $1 \le q \le r < \infty$, $t > 0$ and $u_0 \in L^q_\rho(\mathbb{R}^N)$ we have the following estimates*

$$\begin{cases} \|e^{(\Delta+V)t}u_0\|_{L^r_{\rho^{\frac{r}{q}}}(\mathbb{R}^N)} \le Me^{at}t^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u_0\|_{L^q_\rho(\mathbb{R}^N)} \\ \|e^{(\Delta+V)t}u_0\|_{L^\infty_{\rho^{\frac{1}{q}}}(\mathbb{R}^N)} \le Me^{at}t^{\frac{-N}{2q}}\|u_0\|_{L^q_\rho(\mathbb{R}^N)}. \end{cases}$$

Proof. We prove below that the estimate

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \le Mt^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u(0)\rho\|_{L^q(\mathbb{R}^N)},\ 0 < t \le \tau_0 \tag{3.14}$$

holds for some small time interval $0 < t \le \tau_0$, for some τ_0 and M, depending only on N, σ, and $\|V\|_{L^\sigma_U(\mathbb{R}^N)}$. Once we show this, we extend the estimate for arbitrary t. In fact, if $t \ge \tau_0$ then we decompose it as $t = n\tau_0 + s$ for some $0 \le s < \tau_0$. Iterating n times (3.14) with $q = r$ and denoting $u(t) = e^{(\Delta+V)t}u_0$, we get

$$\begin{aligned} \|u(t)\rho\|_{L^r(\mathbb{R}^N)} &\le M\|u(n\tau_0)\rho\|_{L^r(\mathbb{R}^N)} \le M^2\|u((n-1)\tau_0)\rho\|_{L^r(\mathbb{R}^N)} \\ &\le M^n\|u(\tau_0)\rho\|_{L^r(\mathbb{R}^N)} \le M^{n+1}\tau_0^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u(0)\rho\|_{L^q(\mathbb{R}^N)}, \end{aligned}$$

where we have used (3.14) for time τ_0 in the last step. Hence, since $\tau_0 \le t < (n+1)\tau_0$, $t = n\tau_0 + s$, we get for $t \ge \tau_0$

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \le M^{n+1}\tau_0^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u(0)\rho\|_{L^q(\mathbb{R}^N)} \le M_1e^{at}t^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u(0)\rho\|_{L^q(\mathbb{R}^N)}. \tag{3.15}$$

Putting together the estimates for $0 < t \le \tau_0$ and $t \ge \tau_0$, we get (3.15) for any $t > 0$. Replacing ρ by $\rho^{\frac{1}{q}}$ we get the result.

Now we show the estimate (3.14) for $0 < t \le \tau_0$, and some small $\tau_0 > 0$.

Denoting $u(t) = e^{(\Delta+V)t}u_0$, the semigroup $e^{(\Delta+V)t}u_0$ can be expressed in terms of the variation of constants formula as

$$u(t) = e^{(\Delta+V)t}u_0 = e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}V(x)u(s)\,ds. \tag{3.16}$$

Then, for $1 \le q \le r \le \infty$ and $u_0 \in L^q_\rho(\mathbb{R}^N)$, multiplying (3.16) by $\rho \in R_{\rho_1,\rho_2}$ and taking the norm in $L^r(\mathbb{R}^N)$

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \le \|e^{\Delta t}u_0\,\rho\|_{L^r(\mathbb{R}^N)} + \int_0^t \|e^{\Delta(t-s)}Vu(s)\rho\|_{L^r(\mathbb{R}^N)}\,ds$$

From the smoothing of the linear heat equation in Proposition 3.2, for $f_1(t) = (1+e^{\rho_1^2 Nt} t^{\frac{N}{2}[\frac{1}{r}-\frac{1}{q}+1]})$ and $f_2(t) = (1+e^{\rho_1^2 Nt} t^{\frac{N}{2}[\frac{1}{r}-\frac{1}{\tau}+1]})$ we have

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]} f_1(t)\|u_0\rho\|_{L^q(\mathbb{R}^N)} + \int_0^t M(t-s)^{-\frac{N}{2}[\frac{1}{\tau}-\frac{1}{r}]} f_2(t-s)\|Vu(s)\rho\|_{L^\tau(\mathbb{R}^N)}\, ds$$

for some τ to be chosen.

Since $1-\frac{1}{q}+\frac{1}{r}$ and $1-\frac{1}{\tau}+\frac{1}{r}$ are less than 1, for $t^{\frac{N}{2}} < 1$ we have $|f_1(t)| \leq 2$ and $|f_2(t)| \leq 2$, for $t \in [0, \tau_0]$, for some τ_0 independent of q, r, τ. Thus,

$$\begin{aligned} \|u(t)\rho\|_{L^r(\mathbb{R}^N)} &\leq 2Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u_0\rho\|_{L^q(\mathbb{R}^N)} + 2\int_0^t M(t-s)^{-\frac{N}{2}[\frac{1}{\tau}-\frac{1}{r}]}\|Vu(s)\rho\|_{L^\tau(\mathbb{R}^N)}\, ds \\ &\leq 2Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u_0\rho\|_{L^q(\mathbb{R}^N)} + 2\int_0^t M(t-s)^{-\frac{N}{2}[\frac{1}{\tau}-\frac{1}{r}]}\|V\|_{L^\sigma_U(\mathbb{R}^N)}\|u(s)\rho\|_{L^\eta(\mathbb{R}^N)}\, ds \end{aligned} \tag{3.17}$$

with $\frac{1}{\tau} = \frac{1}{\sigma} + \frac{1}{\eta}$, where we have used Hölder's inequality in the last step. Defining $\eta := \max\{\sigma', r\}$ then we can always find $\tau \geq 1$ such that the above is satisfied.

According to the choice of η we have that the exponent in the integral above (3.17) satisfies: $1 \leq r \leq \sigma'$ then $\eta = \sigma'$ and thus $\tau = 1$. If $r \geq \sigma'$ then $\eta = r$ and $\tau \geq 1$. Therefore

$$\frac{N}{2}(\frac{1}{\tau}-\frac{1}{r}) \leq \begin{cases} \frac{N}{2}(1-\frac{1}{r}) = \frac{N}{2r'} \leq \frac{N}{2\sigma} < 1, & \text{si } 1 \leq r \leq \sigma' \\ \frac{N}{2\sigma} < 1, & \text{si } r \geq \sigma'. \end{cases}$$

The rest of the proof is done in several steps.

Step 1.– Assume $r \geq \sigma'$ and $0 \leq \frac{N}{2}(\frac{1}{q}-\frac{1}{r}) \leq \alpha$, where $\alpha = \frac{N}{2\sigma} < 1$ if $\sigma < \infty$, or $\alpha = \frac{1}{2}$, if $\sigma = \infty$. With this, (3.17) leads to

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq 2Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u_0\rho\|_{L^q(\mathbb{R}^N)} + 2M\|V\|_{L^\sigma_U(\mathbb{R}^N)}\int_0^t (t-s)^{-\frac{N}{2\sigma}}\|u(s)\rho\|_{L^r(\mathbb{R}^N)}\, ds.$$

Using the auxiliary function $h(t) := t^{\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u(t)\rho\|_{L^r(\mathbb{R}^N)}$, we have

$$\begin{aligned} h(t) &\leq 2M\|u_0\rho\|_{L^q(\mathbb{R}^N)} + 2M\|V\|_{L^\sigma_U(\mathbb{R}^N)} t^{\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\int_0^t (t-s)^{-\frac{N}{2\sigma}} h(s) s^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\, ds \\ &\leq 2M\|u_0\rho\|_{L^q(\mathbb{R}^N)} + 2M\|V\|_{L^\sigma_U(\mathbb{R}^N)} t^{\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\sup_{0\leq s\leq t} h(s)\int_0^t (t-s)^{-\frac{N}{2\sigma}} s^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\, ds. \end{aligned} \tag{3.18}$$

Now we change variables as $s = zt$ to obtain

$$\int_0^t (t-s)^{-\frac{N}{2\sigma}} s^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\, ds = t^{1-\frac{N}{2\sigma}} t^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\int_0^1 (1-z)^{-\frac{N}{2\sigma}} z^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\, dz.$$

Using this in (3.18)

$$h(t) \leq 2M\|u_0\rho\|_{L^q(\mathbb{R}^N)} + 2\, M\|V\|_{L^\sigma_U(\mathbb{R}^N)} t^{1-\frac{N}{2\sigma}}\sup_{0\leq s\leq t} h(s)\int_0^1 (1-z)^{-\frac{N}{2\sigma}} z^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\, dz \tag{3.19}$$

The integral above is finite with a bound independent of q and r, since we are assuming $\frac{N}{2}[\frac{1}{q}-\frac{1}{r}] \leq \alpha < 1$. Hence, we can chose $\tau_0 > 0$ depending only on M_0, σ and $\|V\|_{L^\sigma_U(\mathbb{R}^N)}$, such that

$$\tau_0^{1-\frac{N}{2\sigma}} \|V\|_{L^\sigma_U(\mathbb{R}^N)} 2M \int_0^1 (1-z)^{-\frac{N}{2\sigma}} z^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\, dz \leq \frac{1}{2}.$$

Thus, from (3.19), for $0 \leq t \leq \tau_0$ we get

$$h(t) \leq 2M\|u_0\rho\|_{L^q(\mathbb{R}^N)} + \frac{1}{2} \sup_{0\leq s\leq \tau_0} h(s),$$

and then there exists τ_0 such that for $0 \leq t \leq \tau_0$ we obtain

$$h(t) \leq 4M\|u_0\rho\|_{L^q(\mathbb{R}^N)}.$$

This implies

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq 4Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]} \|u_0\rho\|_{L^q(\mathbb{R}^N)} \tag{3.20}$$

as long as $r \geq \sigma'$ and $0 \leq \frac{N}{2}[\frac{1}{q} - \frac{1}{r}] \leq \alpha$, for some τ_0 with $0 \leq t \leq \tau_0$.

Step 2.– Assume $1 \leq r < \sigma'$, $1 \leq q \leq r$.

If $\sigma = \infty$ this case is empty. Hence, we assume below that $\sigma < \infty$.

If $1 \leq r < \sigma'$, we have $\eta = \sigma'$. Thus for $1 \leq q \leq r$, equation (3.17) turns into

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq 2Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]} \|u_0\rho\|_{L^q(\mathbb{R}^N)} + 2M\|V\|_{L^\sigma_U(\mathbb{R}^N)} \int_0^t (t-s)^{-\frac{N}{2\sigma}} \|u(s)\rho\|_{L^{\sigma'}(\mathbb{R}^N)}$$

Considering $r = \sigma'$ in Step 1, we have that for all q with $0 \leq \frac{N}{2}(\frac{1}{q} - \frac{1}{r}) \leq \frac{N}{2\sigma}$, the estimate (3.20) holds with $r = \sigma'$. But this last restriction when $r = \sigma'$ is equivalent to $1 \leq q \leq r$, then we can use the estimate (3.20) for $r = \sigma'$ and $1 \leq q \leq r$, to obtain for some τ_0 and $0 \leq t \leq \tau_0$,

$$\begin{aligned} &\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq 2Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]} \|u_0\rho\|_{L^q(\mathbb{R}^N)} \\ + \quad & 8M^2\|V\|_{L^\sigma_U(\mathbb{R}^N)} \|u_0\rho\|_{L^q(\mathbb{R}^N)} \int_0^t (t-s)^{-\frac{N}{2\sigma}} s^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{\sigma'}]}\, ds. \end{aligned}$$

Changing variables as (3.19) we get

$$\begin{aligned} &\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq 2Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]} \|u_0\rho\|_{L^q(\mathbb{R}^N)} \\ + \quad & 8M^2\|V\|_{L^\sigma_U(\mathbb{R}^N)} \|u_0\rho\|_{L^q(\mathbb{R}^N)} t^{1-\frac{N}{2\sigma}} t^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{\sigma'}]} \int_0^1 (1-z)^{-\frac{N}{2\sigma}} z^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{\sigma'}]}\, dz. \end{aligned}$$

This integral is finite and bounded by a constant C only depending on N and σ. Hence,

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq 2Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]} \|u_0\rho\|_{L^q(\mathbb{R}^N)} \big(1 + \|V\|_{L^\sigma_U(\mathbb{R}^N)} 4MCt^{1-\frac{N}{2\sigma}}\big)$$

Therefore, for some τ_0 depending on σ, N, and $\|V\|_{L^\sigma_U(\mathbb{R}^N)}$

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \leq \hat{M}t^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]} \|u_0\rho\|_{L^q(\mathbb{R}^N)}, \quad 0 < t \leq \tau_0.$$

Step 3.– Summarizing Steps 1 and 2, we have found constants M, τ_0 depending only on N, σ and $\|V\|_{L^\sigma_U(\mathbb{R}^N)}$ such that for all $1 \le q \le r < \infty$ with $\frac{N}{2}[\frac{1}{q} - \frac{1}{r}] \le \alpha$, we have

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \le Mt^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u_0\rho\|_{L^q(\mathbb{R}^N)}, \ \ 0 \le t \le \tau_0. \tag{3.21}$$

Now observe that there exists a natural number K and a partition of the interval $[1,\infty]$, of the form $1 = r_0 < r_1 < r_2 ..... < r_K = \infty$, and such that for each q, r with $r_k \le q \le r \le r_{k+1}$, we have $\frac{N}{2}[\frac{1}{q} - \frac{1}{r}] \le \alpha$.

Therefore, for each $1 \le q \le r < \infty$, there exists k and h such that $q \in [r_k, r_{k+1}]$ and $r \in [r_{k+h}, r_{k+h+1}]$. For each $0 < t < \tau_0$ we take a partition of $[0,t]$ in $h+1$ subintervals of lenght $\frac{t}{h+1}$ and iterate the inequality (3.21) from

$$q \to r_{k+1} \to r_{k+2}...... \to r_{k+h} \to r,$$

$h+1$ times, to obtain

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \le M(\frac{t}{h+1})^{-\frac{N}{2}[\frac{1}{r_{k+h}}-\frac{1}{r}]}\|u(\frac{th}{h+1})\rho\|_{L^{r_{k+h}}(\mathbb{R}^N)}$$

$$\|u(\frac{t}{h+1})\rho\|_{L^{r_{k+h}}(\mathbb{R}^N)} \le M(\frac{t}{h+1})^{-\frac{N}{2}[\frac{1}{r_{k+h-1}}-\frac{1}{r_{k+h}}]}\|u(\frac{(h-1)t}{h+1})\rho\|_{L^{r_{k+h-1}}(\mathbb{R}^N)}$$

$$\vdots$$

$$\|u(\frac{2t}{h+1})\rho\|_{L^{r_{k+2}}(\mathbb{R}^N)} \le M(\frac{t}{h+1})^{-\frac{N}{2}[\frac{1}{r_{k+1}}-\frac{1}{r_{k+2}}]}\|u(\frac{t}{h+1})\rho\|_{L^{r_{k+1}}(\mathbb{R}^N)}$$

$$\|u(\frac{t}{h+1})\rho\|_{L^{r_{k+1}}(\mathbb{R}^N)} \le M(\frac{t}{h+1})^{-\frac{N}{2}[\frac{1}{q}-\frac{1}{r_{k+1}}]}\|u(0)\rho\|_{L^q(\mathbb{R}^N)}.$$

Multiplying term to term, we get

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \le M^{h+1}(\frac{t}{h+1})^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u(0)\rho\|_{L^q(\mathbb{R}^N)}.$$

As $h+1 \le K+1$, we obtain

$$\|u(t)\rho\|_{L^r(\mathbb{R}^N)} \le Lt^{\frac{-N}{2}[\frac{1}{q}-\frac{1}{r}]}\|u(0)\rho\|_{L^q(\mathbb{R}^N)}, \ 0 < t \le \tau_0$$

where $L = M^{K+1}(K+1)^{\frac{N}{2}}$, which depends only on N and $\|V\|_{L^\sigma_U(\mathbb{R}^N)}$.□

Now we want to characterize when the Schrödinger semigroup decays exponentially. In other words, we would like to characterize the best constant in the estimates in Theorem 3.5 when $r = q$.

Let $-A$ be the infinitesimal generator of a C^0 semigroup, $S(t) = e^{-At}$, in a Banach space X.

Definition 3.6. *The exponential type of the semigroup* e^{-At} *is*

$$\sigma_0 := \inf\{\sigma \in \mathbb{R} : \|e^{-At}\|_{\mathcal{L}(X,X)}e^{-\sigma t} \text{ is bounded in } t \in (0,\infty)\}$$

or equivalently

$$\sigma_0 = \inf\{\lambda \in \mathbb{R} : \|e^{-(A+\lambda I)t}\|_{\mathcal{L}(X,X)} \text{ decays exponentially}\}.$$

Now we will study the exponential type of Schrödinger operators of the form $A = -\Delta + m(x)I$, $m \in L^{\sigma}_{U}(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$ in the weighted spaces $L^q_\rho(\mathbb{R}^N)$. For this we first state some properties of these operators in spaces without weight.

First, observe that considering the operator A in the unweighted space $L^q(\mathbb{R}^N)$, the exponential type is independent of q and coincides with the one in $L^2(\mathbb{R}^N)$, which is given by

$$-\Sigma_0(-\Delta + mI) = \inf_{\varphi \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla\varphi(x)|^2\, dx + \int_{\mathbb{R}^N} m(x)\varphi^2(x)\, dx}{\int_{\mathbb{R}^N} |\varphi(x)|^2\, dx}. \tag{3.22}$$

For more details see [27]. As mentoned in the introduction, note that $-\Delta + m(x)I$ is not selfadjoint in $L^2_\rho(\mathbb{R}^N)$, see [14, 15].

Now we state a Lemma that will be very important below; for more details see [7]

Lemma 3.7. *Assume $V \in L^{\sigma}_{U}(\Omega)$ for some $\sigma > \frac{N}{2}$ and it is such that the semigroup $S_V(t)$ decays exponentially with a decay rate $\mu < 0$, with*

$$-\mu < -\Sigma_0(-\Delta + mI). \tag{3.23}$$

Then

(i) There exists $\delta_0 > 0$ such that for every $\lambda \in (0, 1 + \delta_0)$ the semigroup $S_{\lambda V}(t)$ decays exponentially with a decay rate $\mu(\lambda) < 0$, which is continuous in λ.

(ii) There exists $C(\mu)$ such that if $P \in L^p_U(\Omega)$, with $p > \frac{N}{2}$, with negative part such that $\|P^-\|_{L^p_U(\Omega)} \le C(\mu)$, then the semigroup $S_{V+P}(t)$ also decays exponentially.

Now we give a useful consequence of the previous lemma.

Lemma 3.8. *Let $m \in L^{\sigma}_{U}(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, $\sigma > 2$. Then $\Sigma_0(-\Delta + \lambda mI)$ is a continuous functions of λ.*

Proof. Let $\lambda_0 \in \mathbb{R}$ and $A = -\Delta + \lambda_0 m(x)I$. If σ is such that $\frac{\|e^{-At}\|_{\mathcal{L}(L^2,L^2)}}{e^{\sigma t}} \le K$, for every $0 < t < \infty$, then we have that $\|e^{-(A+\sigma I+\varepsilon I)t}\|_{\mathcal{L}(L^2,L^2)}$ decays exponentially for every $\varepsilon > 0$.

Consider the potential $V(x) = \lambda_0 m(x) + \sigma + \varepsilon$, by Lemma 3.7 there exists δ_0 such that $\Sigma_0(-\Delta + \mu V I)$ is continuous for $\mu \in (0, 1 + \delta_0)$. Moreover, we have

$$-\Sigma_0(-\Delta + \mu V I) \;=\; \Sigma_0(-\Delta + \mu\lambda_0 m) + \mu(\sigma + \varepsilon)$$

Thus, we have that $\Sigma_0(-\Delta + \lambda mI)$ with $\lambda = \mu\lambda_0$ is also a continuous function in a neighborhood of λ_0.□

Observe that considering the operator A in the spaces $L^q_U(\mathbb{R}^N)$, it was proved in [6] that the exponential type is independent of q and coincides with the one in $L^r(\mathbb{R}^N)$. That is, it is given by $\sigma_0(L^2)$, in (3.22).

Now in the space $L^q_\rho(\mathbb{R}^N)$ we do not know if such property is true in general, although we prove the following

Proposition 3.9. *Assume $m \in L^{\sigma}_{U}(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, $\sigma > q > 1$ and $\rho \in R_{\rho_1,\rho_2}$.*
i) For sufficiently small ε we have

$$\sigma_0(-\Delta + mI, L^q_{\rho_\varepsilon}(\mathbb{R}^N)) = \Sigma_0(-\Delta + mI)$$

where $\rho_\varepsilon(x) = \rho(\varepsilon x)$.
ii) If for every small $\varepsilon > 0$

$$\lim_{|x|\to\infty} \frac{\rho(\varepsilon x)}{\rho(x)} = C_\varepsilon > 0$$

then

$$\sigma_0(-\Delta + mI, L^q_\rho(\mathbb{R}^N)) = \Sigma_0(-\Delta + mI)$$

Proof. Denote $A = -\Delta + m(x)I$. Given $\delta \in \mathbb{R}$, consider the isomorphism in Proposition 2.9. Then we have

$$\begin{array}{ccc} L^q_\rho(\mathbb{R}^N) & \xrightarrow{A+\delta I} & L^q_\rho(\mathbb{R}^N) \\ {\scriptstyle J_q}\downarrow & & \uparrow{\scriptstyle J_q^{-1}} \\ L^q(\mathbb{R}^N) & \xrightarrow{L(\delta)} & L^q(\mathbb{R}^N) \end{array}$$

with

$$L(\delta) := -\Delta + \frac{2}{q}\frac{\nabla\rho}{\rho}\nabla + [\frac{1}{q}\frac{\Delta\rho}{\rho} - \frac{(1+\frac{1}{q})}{q}\frac{|\nabla\rho|^2}{\rho^2}]I + m(x)I + \delta I$$

as in (3.2) and we have $\|e^{-(A+\delta I)t}\|_{\mathcal{L}(L^q_\rho(\mathbb{R}^N))} = \|e^{-Lt}\|_{\mathcal{L}(L^q(\mathbb{R}^N))}$.

The construction above for the weight $\rho_\varepsilon(x)$ gives

$$L(\delta) := A + \frac{2}{q}\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon}\nabla + [\frac{1}{q}\frac{\Delta\rho_\varepsilon}{\rho_\varepsilon} - \frac{(1+\frac{1}{q})}{q}\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon^2}]I + \delta I.$$

Thus, we define $V^1_\varepsilon(x) = \frac{2}{q}\frac{\nabla\rho_\varepsilon}{\rho_\varepsilon}$ and $V^2_\varepsilon(x) = \frac{1}{q}\frac{\Delta\rho_\varepsilon}{\rho_\varepsilon} - \frac{(1+\frac{1}{q})}{q}\frac{|\nabla\rho_\varepsilon|^2}{\rho_\varepsilon^2}$. By Lemma 2.3 we have $|V^1_\varepsilon(x)| \le C_1\varepsilon$ and $|V^2_\varepsilon(x)| \le C_2\varepsilon^2$.

Denoting $L_1 := A + V^1_\varepsilon(x)\nabla + \delta I$, we can express L as

$$L(\delta) = L_1 + V^2_\varepsilon(x)I$$

and the perturbation $P_\varepsilon : L^q(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ defined by $P_\varepsilon(u) = V^2_\varepsilon(x)u$, satisfies

$$\|P_\varepsilon u\|_{L^q(\mathbb{R}^N)} \le C\varepsilon^2\|u\|_{L^q(\mathbb{R}^N)}.$$

By Lemma 3.10 below, $e^{-L(\delta)t}$ and e^{-L_1t} have the same exponential decay in $L^q(\mathbb{R}^N)$ for ε small enough.

Again we have $L_1 = A + V^1_\varepsilon(x)\nabla + \delta I$ and now the perturbation $Q_\varepsilon : H^{1,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ defined by $Q_\varepsilon(u) = V^1_\varepsilon(x)\nabla u$, satisfies

$$\|Q_\varepsilon u\|_{L^q(\mathbb{R}^N)} \le C_0\varepsilon\|u\|_{H^{1,q}(\mathbb{R}^N)}.$$

Again by Lemma 3.10, e^{-L_1t} and $e^{-(A+\delta)t}$ have the same exponential decay in $L^q(\mathbb{R}^N)$ for ε small enough. This gives part i).

Part ii) follows from the above and Remark 2.7. □

Now we prove the lemma used above.

Lemma 3.10. *Assume A is a sectorial operator in a Banach space X with fractional power spaces X^α. Assume for $\varepsilon > 0$, $P_\varepsilon : X^\alpha \to X$, with $\alpha \in [0,1)$, is a linear perturbation such that*

$$\|P_\varepsilon\|_{\mathcal{L}(X^\alpha,X)} \leq \varepsilon C.$$

Then if ε is small enough, the semigroup generated by $-A$, that is, e^{-At}, decays exponentially in X if and only if the semigroup $e^{-(A+P_\varepsilon)t}$ decays exponentially in X.

Proof. Consider $u(t) := e^{-(A+P_\varepsilon)t}u_0$ the solution of

$$\begin{cases} u_t + Au = -P_\varepsilon(u) \\ u(0) = u_0 \in X. \end{cases}$$

Using the variation of constants formula, we have for $t > 0$

$$u(t, u_0) = e^{-At}u_0 - \int_0^t e^{-A(t-s)} P_\varepsilon(u(s))\, ds$$

Taking the norm of X^α, using the smoothing of the semigroup and the assumption on the perturbation, we get that for some $\lambda > 0$

$$\|u(t)\|_{X^\alpha} \leq \frac{Me^{-\lambda t}}{t^\alpha}\|u_0\|_X + \varepsilon CM \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\|u(s)\|_{X^\alpha}\, ds.$$

From this, for $t > 0$

$$t^\alpha e^{\lambda t}\|u(t)\|_{X^\alpha} \leq M\|u_0\|_X + \varepsilon CMt^\alpha \int_0^t (t-s)^{-\alpha}s^{-\alpha}[s^\alpha e^{\lambda s}\|u(s)\|_{X^\alpha}]\, ds.$$

Denoting $K(t) := \sup_{s\in[0,t]} s^\alpha e^{\lambda s}\|u(s)\|_{X^\alpha}$, then for any fixed $T > 0$ and for all $0 < t < T$,

$$t^\alpha e^{\lambda t}\|u(t)\|_{X^\alpha} \leq M\|u_0\|_X + \varepsilon CMK(T)t^\alpha \int_0^t (t-s)^{-\alpha}s^{-\alpha}\, ds. \tag{3.24}$$

Changing variables as $s := ty$

$$\int_0^t (t-s)^{-\alpha}s^{-\alpha}\, ds = t^{1-2\alpha}\int_0^1 (1-y)^{-\alpha}y^{-\alpha}\, dy := \beta t^{1-2\alpha}$$

we get in (3.24)

$$t^\alpha e^{\lambda t}\|u(t)\|_{X^\alpha} \leq M\|u_0\|_X + \varepsilon CMt^{1-\alpha}K(T)\beta.$$

Now taking the sup in $t \in (0,T)$

$$K(T) \leq M\|u_0\| + \varepsilon CMT^{1-\alpha}K(T)\beta.$$

Let $C_0 := CM\beta$, if $\varepsilon C_0 T^{1-\alpha} < 1$ then $K(T) \leq \frac{M}{1-\varepsilon C_0 T^{1-\alpha}}\|u_0\|$. In particular, for $t \in (0,T]$, $t^\alpha e^{\lambda t}\|u(t)\|_{X^\alpha} \leq \frac{M\|u_0\|}{1-\varepsilon C_0 T^{1-\alpha}}$. Hence

$$\|u(T)\|_X \leq C\|u(T)\|_{X^\alpha} \leq \frac{MC}{1-\varepsilon C_0 T^{1-\alpha}}\frac{e^{-\lambda T}}{T^\alpha}\|u_0\|_X.$$

Let T be such that $\alpha_0 := \frac{MCe^{-\lambda T}}{T^\alpha} < \frac{1}{4}$. Taking ε_0, such that $\varepsilon_0 C_0 T^{1-\alpha} < \frac{1}{2}$, then for every ε, with $\varepsilon \in (0, \varepsilon_0)$ we get $\|u(T)\|_X \leq 2\alpha_0 \|u_0\|_X$, for every $u_0 \in X$, which implies

$$\|e^{-(A+P_\varepsilon)T}\| \leq 2\alpha_0 < \frac{1}{2} \text{ for every } \varepsilon \in (0, \varepsilon_0).$$

Iterating the semigroup we get that $\|e^{-(A+P_\varepsilon)t}\| \to 0$ exponentially.

The converse is now immediate.□

Another property that will be needed below is the following consequence of the Lumer–Phillips Theorem, see [23], which holds in Hilbert spaces.

Lemma 3.11. *Assume $-A$ is the generator of a C_0 semigroup in a real Hilbert space H with scalar product $<,>$. Then they are equivalent*

(i) $< Au, u > \geq -\mu |u|^2$, for every $u \in D(A)$, $\mu \in \mathbb{R}$.

(ii) $\|e^{-At}\|_{\mathcal{L}(H,H)} \leq e^{\mu t}$

Proof. (i) holds iff $A + \mu I$ is dissipative, which by the Lummer Phillips Theorem, holds iff $-A - \mu I$ generates a contraction semigroup, $\|e^{-(A+\mu I)t}\| \leq 1$, which is equivalent to (ii).□

Remark 3.12. (i) *In the conditions of the Lemma, if we take an equivalent norm in H, then $\|e^{-At}\| \leq Me^{\mu t}$.*

(ii) *Hence, if we define*

$$-\mu := \inf_{u \in D(A)} \frac{< Au, u >}{|u|^2}.$$

Then for all σ such that $-\sigma < -\mu$, we have $< Au, u > \geq -\sigma |u|^2$ and from here we obtain $\|e^{-At}\| \leq e^{\sigma t}$. Then by definition of exponential type we get $\sigma \geq \sigma_0$, that is

$$\mu \geq \sigma_0.$$

For the operator $A = -\Delta + m(x)I$, $m \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, $\sigma > 2$ in $L^2_\rho(\mathbb{R}^N)$, we have the following

Proposition 3.13. *Consider the operator $A = -\Delta + m(x)I$, $m \in L^\sigma_U(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, $\sigma > 2$ in $L^2_\rho(\mathbb{R}^N)$, for an arbitrary weight $\rho \in R_{\rho_1,\rho_2}$. Then*

(1) *for every $0 < \delta < 1$, there exists a constant $C_\delta > 0$ such that*

$$\begin{aligned}(1-\delta) \inf_{u \in H^1_\rho(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \rho + \int_{\mathbb{R}^N} \frac{m}{1-\delta} |u|^2 \rho}{|u|^2_{L^2_\rho(\mathbb{R}^N)}} - C_\delta &\leq \inf_{u \in H^1_\rho(\mathbb{R}^N)} \frac{< Au, u >_{L^2_\rho(\mathbb{R}^N)}}{|u|^2_{L^2_\rho(\mathbb{R}^N)}} \\ &\leq (1+\delta) \inf_{u \in H^1_\rho(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \rho + \int_{\mathbb{R}^N} \frac{m}{1+\delta} |u|^2 \rho}{|u|^2_{L^2_\rho(\mathbb{R}^N)}} + C_\delta\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}-(1-\delta)\Sigma_0(-\Delta + \frac{m}{1-\delta}) - C_\delta &\leq \inf_{u \in H^1_\rho(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \rho + \int_{\mathbb{R}^N} m(x) |u|^2 \rho}{\int_{\mathbb{R}^N} |u|^2 \rho} \\ &\leq -(1+\delta)\Sigma_0(-\Delta + \frac{m}{1+\delta}) + C_\delta\end{aligned} \tag{3.26}$$

and $C_\delta \to \infty$ as $\delta \to 0$.

(2) *Considering the weight $\rho_\varepsilon(x) = \rho(\varepsilon x)$,*

$$-\Sigma_0(-\Delta + mI) = \lim_{\varepsilon\to 0}\left[\inf_{u\in H^1_{\rho_\varepsilon}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N}|\nabla u|^2\rho_\varepsilon + \int_{\mathbb{R}^N} m(x)u^2\rho_\varepsilon}{\int_{\mathbb{R}^N}|u|^2\rho_\varepsilon}\right] = \lim_{\varepsilon\to 0}\left[\inf_{u\in D(A)} \frac{< Au, u >_{\rho_\varepsilon}}{|u|^2_{\rho_\varepsilon}}\right].$$

Proof. To prove (3.25), observe that

$$< (-\Delta u + m(x)I)u, u >_{L^2_\rho(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u\, \nabla u \nabla\rho + \int_{\mathbb{R}^N} |\nabla u|^2\rho + \int_{\mathbb{R}^N} m(x)|u|^2\rho. \qquad (3.27)$$

By Young's inequality, for every $\delta > 0$ there exists C_δ with $C_\delta \to 0$ as $\delta \to 0$, such that

$$\left|\int_{\mathbb{R}^N} u\, \nabla u\nabla\rho\right| \le \delta\int_{\mathbb{R}^N}|\nabla u|^2\rho + C_\delta\int_{\mathbb{R}^N}|u|^2\rho \qquad (3.28)$$

and substituting (3.28) in (3.27) and dividing by $|u|^2_{L^2_\rho(\mathbb{R}^N)}$ we get

$$(1-\delta)\frac{\left[\int_{\mathbb{R}^N}|\nabla u|^2\rho + \int_{\mathbb{R}^N}\frac{m(x)}{1-\delta}|u|^2\rho\right]}{|u|^2_{L^2_\rho(\mathbb{R}^N)}} - C_\delta \le \frac{< Au, u >}{|u|^2_{L^2_\rho(\mathbb{R}^N)}} \le$$
$$(1+\delta)\frac{\left[\int_{\mathbb{R}^N}|\nabla u|^2\rho + \int_{\mathbb{R}^N}\frac{m(x)}{1+\delta}|u|^2\rho\right]}{|u|^2_{L^2_\rho(\mathbb{R}^N)}} + C_\delta.$$

Taking the inf $H^1_\rho(\mathbb{R}^N)$ we get (3.25).

Now we prove (3.26). We take $v = u\rho^{\frac{1}{2}}$ and we have that $\nabla u = \frac{-1}{2}v\rho^{\frac{-3}{2}}\nabla\rho + \rho^{\frac{-1}{2}}\nabla v$. Hence

$$\int_{\mathbb{R}^N}|\nabla u|^2\rho + \int_{\mathbb{R}^N} m(x)|u|^2\rho = \int_{\mathbb{R}^N}|\nabla v|^2 + \int_{\mathbb{R}^N} m(x)|v|^2$$
$$+ \int_{\mathbb{R}^N}\frac{1}{4}|v|^2\frac{|\nabla\rho|^2}{\rho^2} - 2\int_{\mathbb{R}^N} v\nabla v\frac{\nabla\rho}{\rho}. \qquad (3.29)$$

Now Cauchy-Schwartz's and Young's inequalities give, for every $\delta > 0$

$$\left|2\int_{\mathbb{R}^N} v\nabla v\frac{\nabla\rho}{\rho}\right| \le 2\int_{\mathbb{R}^N} v|\nabla v|C_1 \le \delta\int_{\mathbb{R}^N}|\nabla v|^2 + C_\delta\int_{\mathbb{R}^N}|v|^2 \qquad (3.30)$$

and

$$\left|\frac{1}{4}\int_{\mathbb{R}^N}|v|^2|\frac{\nabla\rho}{\rho}|^2\,dx\right| \le \frac{C_0}{4}\int_{\mathbb{R}^N}|v|^2\,dx. \qquad (3.31)$$

Replacing (3.30), (3.31) in (3.29)

$$(1-\delta)\frac{[\int_{\mathbb{R}^N}|\nabla v|^2 + \int_{\mathbb{R}^N}\frac{m(x)}{1-\delta}|v|^2]}{|v|^2_{L^2(\mathbb{R}^N)}} + (\frac{-1}{4}C_0 - C_\delta) \le \frac{\int_{\mathbb{R}^N}|\nabla u|^2\rho + \int_{\mathbb{R}^N} m(x)|u|^2\rho}{|u|^2_{L^2_\rho(\mathbb{R}^N)}}$$
$$\le (1+\delta)\frac{\int_{\mathbb{R}^N}|\nabla v|^2 + \int_{\mathbb{R}^N}\frac{m(x)}{1+\delta}|v|^2}{|v|^2_{L^2(\mathbb{R}^N)}} + (\frac{C_0}{4} + C_\delta) \qquad (3.32)$$

Taking the inf and considering that the set of $u \in H^1_\rho(\mathbb{R}^N)$ is equivalent to the set of $v \in H^1(\mathbb{R}^N)$, where $v = u\rho^{\frac{1}{2}}$, we have

$$
\begin{aligned}
-(1-\delta)\Sigma_0(-\Delta + \frac{m}{1-\delta}) \quad + \quad (\frac{-1}{4}C_0 - C_\delta) \le \inf_{u\in H^1_\rho(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\rho + \int_{\mathbb{R}^N} m(x)|u|^2\rho}{|u|^2_{L^2_\rho(\mathbb{R}^N)}} \\
\le \quad -(1+\delta)\Sigma_0(-\Delta + \frac{m}{1+\delta}) + (\frac{C_0}{4} + C_\delta) \qquad (3.33)
\end{aligned}
$$

which proves (3.26).

Now we prove part (2). Again setting $v = u\rho_\varepsilon^{\frac{1}{2}}$ we get (3.29) with weight ρ_ε. Now, instead of (3.30), by Lemma 2.3 we get

$$
\left| 2\int_{\mathbb{R}^N} v\ \nabla v \frac{\nabla \rho_\varepsilon}{\rho_\varepsilon} \right| \le \varepsilon \int_{\mathbb{R}^N} |v||\nabla v| \le \varepsilon \int_{\mathbb{R}^N} |v|^2\ dx + \varepsilon \int_{\mathbb{R}^N} |\nabla v|^2
$$

and instead of (3.31)

$$
\left| \int_{\mathbb{R}^N} |v|^2 |\frac{\nabla \rho_\varepsilon}{\rho_\varepsilon}|^2 \right| \le \varepsilon^2 \int_{\mathbb{R}^N} |v|^2.
$$

Thus, instead of (3.32) we get

$$
\begin{aligned}
(1-\varepsilon)\left[\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} \frac{m(x)}{1-\varepsilon}|v|^2\right] + (-\frac{\varepsilon^2}{4} - \varepsilon) \le \int_{\mathbb{R}^N} |\nabla u|^2\rho_\varepsilon + \int_{\mathbb{R}^N} m(x)|u|^2\rho_\varepsilon \\
\le (1+\varepsilon)\left[\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} \frac{m(x)}{1+\varepsilon}|v|^2\right] + (\frac{\varepsilon^2}{4} + \varepsilon).
\end{aligned}
$$

Taking the inf, as in the first part of the Theorem, instead of (3.33) we end up with

$$
\begin{aligned}
-(1-\varepsilon)\Sigma_0(-\Delta + \frac{m}{1-\varepsilon}I) + (-\frac{\varepsilon^2}{4} - \varepsilon) \le (1-\varepsilon) \inf_{v\in H^1(\mathbb{R}^N)} \frac{[\int_{\mathbb{R}^N} |\nabla v|^2 + \int_{\mathbb{R}^N} \frac{m(x)}{1-\varepsilon}|v|^2]}{\int_{\mathbb{R}^N} |v|^2} \\
+(-\frac{\varepsilon^2}{4} - \varepsilon) \le \inf_{H^1_{\rho_\varepsilon}(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\rho_\varepsilon + \int_{\mathbb{R}^N} m(x)|u|^2\rho_\varepsilon}{\int_{\mathbb{R}^N} |u|^2\rho_\varepsilon} \le -(1+\varepsilon)\Sigma_0(-\Delta + \frac{mI}{1+\varepsilon}) + (\frac{\varepsilon^2}{4} + \varepsilon) \\
(3.34)
\end{aligned}
$$

and taking limits as $\varepsilon \to 0$, using the continuity of the exponential type $\Sigma_0(-\Delta + \lambda mI)$ given in Lemma 3.8 we obtain the first part of (2).

For the second part of (2) observe that

$$
< Au, u >_{L^2_\rho(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\nabla u|^2\rho_\varepsilon + \int_{\mathbb{R}^N} m(x)|u|^2\rho_\varepsilon + \int_{\mathbb{R}^N} u\nabla u \nabla \rho_\varepsilon.
$$

By Lemma 2.3 and Cauchy Schwartz inequality, the right hand side above is bounded by

$$
\left| \int_{\mathbb{R}^N} u\nabla u\nabla\rho_\varepsilon\ dx \right| \le \varepsilon\Big(\int_{\mathbb{R}^N} |u|^2\rho_\varepsilon + \int_{\mathbb{R}^N} |\nabla u|^2\rho_\varepsilon \Big).
$$

From this inequality we get

$$(1-\varepsilon)\inf_{u\in H^1_{\rho_\varepsilon}(\mathbb{R}^N)}\frac{[\int_{\mathbb{R}^N}|\nabla u|^2\rho_\varepsilon+\int_{\mathbb{R}^N}\frac{m(x)}{1-\varepsilon}|u|^2\ \rho_\varepsilon]}{\int_{\mathbb{R}^N}|u|^2\ \rho_\varepsilon}-\varepsilon$$

$$\leq \inf_{u\in H^1_{\rho_\varepsilon}(\mathbb{R}^N)}\frac{<Au,u>_{L^2_\rho(\mathbb{R}^N)}}{\int_{\mathbb{R}^N}|u|^2\rho_\varepsilon}\leq$$

$$\leq (1+\varepsilon)\inf_{u\in H^1_{\rho_\varepsilon}(\mathbb{R}^N)}\frac{[\int_{\mathbb{R}^N}|\nabla u|^2\rho_\varepsilon+\int_{\mathbb{R}^N}\frac{m(x)}{1+\varepsilon}|u|^2\rho_\varepsilon]}{\int_{\mathbb{R}^N}|u|^2\rho_\varepsilon}+\varepsilon.$$

Using (3.34)

$$\begin{aligned}
-(1-\varepsilon)^2(1-\varepsilon)\Sigma_0(-\Delta &+ \frac{m}{(1-\varepsilon)^2}I)+(-\frac{\varepsilon^2}{4}-\varepsilon)(1-\varepsilon)-\varepsilon\\
&\leq (1-\varepsilon)\inf_{u\in H^1_{\rho_\varepsilon}(\mathbb{R}^N)}\frac{[\int_{\mathbb{R}^N}|\nabla u|^2\rho_\varepsilon+\int_{\mathbb{R}^N}\frac{m(x)}{1-\varepsilon}|u|^2\ \rho_\varepsilon]}{\int_{\mathbb{R}^N}|u|^2\ \rho_\varepsilon}-\varepsilon\\
&\leq \inf\frac{<Au,u>_{L^2_\rho(\mathbb{R}^N)}}{\int_{\mathbb{R}^N}|u|^2\rho_\varepsilon}\leq\\
&\leq (1+\varepsilon)\inf_{u\in H^1_{\rho_\varepsilon}(\mathbb{R}^N)}\frac{[\int_{\mathbb{R}^N}|\nabla u|^2\rho_\varepsilon+\int_{\mathbb{R}^N}+\frac{m(x)}{1+\varepsilon}|u|^2\rho_\varepsilon]}{\int_{\mathbb{R}^N}|u|^2\rho_\varepsilon}+\varepsilon\\
&= -(1+\varepsilon)^2\Sigma_0(-\Delta+\frac{m}{(1+\varepsilon)^2}I)+(\frac{\varepsilon^2}{4}+\varepsilon)(1+\varepsilon)+\varepsilon.
\end{aligned}$$

Taking limits as $\varepsilon\to 0$ and considering the continuity of the exponential type of the operator $-\Delta+\lambda m(x)$ as in Lemma 3.8 we get the result.□

Now we give the following consequence of the previous results.

Corollary 3.14. *Assume $m\in L^\sigma_U(\mathbb{R}^N)$, $\sigma>\frac{N}{2}$ and $\sigma>2$, $\rho\in R_{\rho_1,\rho_2}$. Then there exists $\mu=\mu(m)\in\mathbb{R}$ depending only on ρ_1,ρ_2 but independent of the particular weight ρ, such that*
(i) $\int_{\mathbb{R}^N}|\nabla\phi(x)|^2\rho(x)+\int_{\mathbb{R}^N}m(x)\phi(x)^2\rho(x)\geq\mu\int_{\mathbb{R}^N}\phi(x)^2\rho(x)$,
(ii) $\lambda\to\mu(\lambda m)$ is continuous.

Proof. From (3.26), using Proposition 3.13, taking $\delta=\frac{1}{2}$, we get that we can take $\mu(m)=\frac{-1}{2}\Sigma_0(-\Delta+2mI)-C$. This gives $\mu(\lambda m)=C\Sigma_0(-\Delta+2\lambda mI)-C$, but since $-\Sigma_0(-\Delta+\lambda m)$ is continuous in λ, we get the result.□

4 Local and global existence and uniform bounds of solutions for nonlinear problems.

Now we study the local and global existence and regularity of the following reaction diffusion equation in the space $X=L^q_\rho(\mathbb{R}^N)$, with $1<q<\infty$,

$$\begin{cases} u_t-\Delta u=f(x,u), \text{ with } x\in\mathbb{R}^N, t>0\\ u(0)=u_0,\end{cases}\tag{4.1}$$

where $\rho\in R_{\rho_1,\rho_2}$.

Theorem 4.1. *Let $1 < q < \infty$. Consider problem (4.1) with $u_0 \in H^{1,q}_\rho(\mathbb{R}^N)$, where $\rho \in R_{\rho_1,\rho_2}$. We assume the nonlinear part can be written as*

$$f(x,u) = g(x) + m(x)u + f_0(x,u)$$

in such a way that $g \in L^q_\rho(\mathbb{R}^N)$, $m \in L^\sigma_U(\mathbb{R}^N)$ with $\sigma > \frac{N}{2}$, $\sigma > q$, and f_0 satisfies,

$$\begin{cases} f_0(x,0) = 0, \quad \frac{\partial f_0}{\partial s}(x,0) = 0 \\ |\frac{\partial}{\partial s} f_0(x,s)| \le C(1 + |s\rho^{\frac{1}{q}}(x)|^{r-1}), \ \ s \in \mathbb{R}, \ x \in \mathbb{R}^N \end{cases} \tag{4.2}$$

$$\text{with } 1 \le r \begin{cases} < \infty, & \text{if } N \le q \\ \le \frac{N}{N-q}, & \text{if } N > q. \end{cases} \tag{4.3}$$

Then there exists a unique solution of (4.1) given by the variations of constants formula,

$$u(t,u_0) = S(t)u_0 + \int_0^t S(t-s)(g + f_0^e(u(s)))\, ds, \qquad t \in [0,\tau_0) \tag{4.4}$$

where $S(t)$ denotes the linear analytic semigroup genated by $\Delta + m(x)I$ in $L^q_\rho(\mathbb{R}^N)$, that is $S(t) = e^{(\Delta+m(x)I)t}$, and $[0,\tau_0)$ is the maximal existence interval of the solution. Moreover the solution satisfies, for every $\gamma \in [0,1)$

$$u(.,u_0) \in C([0,\tau_0), H^{1,q}_\rho(\mathbb{R}^N)) \cap C((0,\tau_0), H^{2,q}_\rho(\mathbb{R}^N)) \cap C^1((0,\tau_0), H^{2\gamma,q}_\rho(\mathbb{R}^N)).$$

Here $H^{2\gamma,q}_\rho(\mathbb{R}^N)$, with $\gamma \in [0,1)$, denotes the fractional power spaces of the operator $-\Delta$ in $L^q_\rho(\mathbb{R}^N)$.

Proof. We will use Theorem 3.3.3 in [20], page 54 in the base space $X = L^q_\rho(\mathbb{R}^N)$ and with $X^{\frac{1}{2}} = H^{1,q}_\rho(\mathbb{R}^N)$. Since $g \in L^q_\rho(\mathbb{R}^N)$, it is enough to prove that $f_0^e : H^{1,q}_\rho(\mathbb{R}^N) \to L^q_\rho(\mathbb{R}^N)$ is Lipschitz on bounded sets.

First we are going to prove that the Nemitcky operator associated to f_0 transforms $H^{1,q}_\rho(\mathbb{R}^N)$ into $L^q_\rho(\mathbb{R}^N)$. Thus, if $u \in H^{1,q}_\rho(\mathbb{R}^N)$, using the weighted Sobolev embeddings in Lemma 2.12, $u \in L^p_{\rho^{\frac{p}{q}}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |u(x)|^p \rho(x)^{\frac{p}{q}}\, dx \le \|u\|^p_{H^{1,q}_\rho(\mathbb{R}^N)}, \quad \text{with } q \le p \begin{cases} < \infty, \text{ if } N \le q \\ \le \frac{qN}{N-q}, \ \text{ if } 1 \le q < N. \end{cases} \tag{4.5}$$

Now, by (4.2) we have,

$$\int_{\mathbb{R}^N} |f_0(x,u(x))|^q \rho(x)\, dx \le C \int_{\mathbb{R}^N} (1 + |u(x)|^{q(r-1)} \rho(x)^{r-1}) |u(x)|^q \rho(x)\, dx$$

that is

$$\int_{\mathbb{R}^N} |f_0(x,u(x))|^q \rho(x)\, dx \le C \int_{\mathbb{R}^N} |u(x)|^q \rho(x)\, dx + C \int_{\mathbb{R}^N} |u(x)|^{qr} \rho(x)^r\, dx. \tag{4.6}$$

The first integral in the right hand side is finite since $u \in L^q_\rho(\mathbb{R}^N)$, while the second one is also finite by (4.5) and (4.3).

Now we prove that the Nemitcky operator is Lipschitz on bounded sets. Let $u,\, v \in H^{1,q}_\rho(\mathbb{R}^N)$ with $\|u\|_{H^{1,q}_\rho(\mathbb{R}^N)},\ \|v\|_{H^{1,q}_\rho(\mathbb{R}^N)} \leq R$ for some $R > 0$. Using again (4.2),

$$|f_0(x,u(x)) - f_0(x,v(x))| \leq C(1 + |u(x)|^{r-1}\rho(x)^{\frac{r-1}{q}} + |v(x)|^{r-1}\rho(x)^{\frac{r-1}{q}})|v(x) - u(x)|$$

and then

$$\|f_0^e(u) - f_0^e(v)\|^q_{L^q_\rho(\mathbb{R}^N)} \leq C_1 \int_{\mathbb{R}^N} |v(x) - u(x)|^q \rho(x)\, dx +$$

$$+ C_1 \int_{\mathbb{R}^N} \rho(x)^{r-1}(|u(x)|^{q(r-1)} + |v(x)|^{q(r-1)})|v(x) - u(x)|^q \rho(x)\, dx.$$

In the second integral we use Hölder's inequality, with $1 < s, s' < \infty$ such that $\frac{1}{s} + \frac{1}{s'} = 1$,

$$\int_{\mathbb{R}^N} \rho(x)^{r-1}(|u(x)|^{q(r-1)} + |v(x)|^{q(r-1)})|v(x) - u(x)|^q \rho(x)\, dx \leq$$

$$\leq C_2 \left[\int_{\mathbb{R}^N} \rho(x)^{s(r-1)}|u(x)|^{qs(r-1)} + \int_{\mathbb{R}^N} \rho(x)^{s(r-1)}|v(x)|^{qs(r-1)} \right]^{\frac{1}{s}} . \|(v-u)\rho^{\frac{1}{q}}\|^q_{L^{qs'}(\mathbb{R}^N)}$$

$$= C_2 \left[\|u\|^{qs(r-1)}_{L^{qs(r-1)}_{\rho^{s(r-1)}}} + \|v\|^{qs(r-1)}_{L^{qs(r-1)}_{\rho^{s(r-1)}}} \right]^{\frac{1}{s}} \|v-u\|^q_{L^{qs'}_{\rho^{s'}}(\mathbb{R}^N)}.$$

To use (4.5) we need $q \leq qs(r-1) \leq \frac{qN}{N-q}$ and $q \leq qs' \leq \frac{qN}{N-q}$, if $1 \leq q < N$. Hence, if $1 \leq q < N$, we take $s' = \frac{N}{N-q}$ and then $s = \frac{N}{q}$ and $qs(r-1) \leq \frac{qN}{N-q}$, since $r \leq \frac{N}{N-q}$. On the other hand, if $N \leq q$, we take s arbitrary. In such a case we get

$$\|f_0^e(u) - f_0^e(v)\|^q_{L^q_\rho(\mathbb{R}^N)} \leq C_3\|v-u\|^q_{H^{1,q}_\rho(\mathbb{R}^N)} + C_3 \left[\|u\|^{qs(r-1)}_{H^{1,q}_\rho(\mathbb{R}^N)} + \|v\|^{qs(r-1)}_{H^{1,q}_\rho(\mathbb{R}^N)} \right]^{\frac{1}{s}} \|v-u\|^q_{H^{1,q}_\rho(\mathbb{R}^N)}.$$

Since, $u, v \in H^{1,q}_\rho(\mathbb{R}^N)$, with $\|u\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|v\|_{H^{1,q}_\rho(\mathbb{R}^N)} \leq R$, we get

$$\|f_0^e(u) - f_0^e(v)\|_{L^q_\rho(\mathbb{R}^N)} \leq C_4(R)\|v-u\|_{H^{1,q}_\rho(\mathbb{R}^N)}. \square$$

Remark 4.2. *In the present work we are not focussing on the issue of critical exponents and optimal growth condition on the nonlinearities to obtain a local well posed problem and actually, the exponents from (4.3) are not optimal. As a matter of fact, using the arguments developed in [4] and further extended in [5], an existence and uniqueness theorem can be obtaind when the nonlinearity f_0 satisfies (4.2) where $1 \leq r < \infty$ if $N \leq q$ and $1 \leq r \leq \frac{N+q}{N-q}$ if $N > q$.*

Now we will show some dissipativity conditions that guarantee the global existence of solutions of (4.1).

Theorem 4.3. *Assume $1 < q < \infty$. Under the assumptions of Theorem 4.1, assume that there exists $C \in L^\sigma_U(\mathbb{R}^N)$ with $\sigma > \frac{N}{2}, \sigma > q$, and $0 \leq D \in L^q_\rho(\mathbb{R}^N)$ such that the nonlinear term satisfies*

$$f(x,s)s \leq C(x)|s|^2 + D(x)|s|, \quad \text{for all} \quad s \in \mathbb{R},\ x \in \mathbb{R}^N. \tag{4.7}$$

Then the solution of (4.1) with initial condition $u_0 \in H^{1,q}_\rho(\mathbb{R}^N)$ *is globally defined.*

Thus, (4.1) defines a nonlinear semigroup $\{T(t)\}_{t\geq 0}$,

$$T(t) : H^{1,q}_\rho(\mathbb{R}^N) \to H^{1,q}_\rho(\mathbb{R}^N) \tag{4.8}$$

as $T(t)u_0 := u(t)$*, where* $u(t)$ *is the solution of (4.1) with data* u_0*.*

Proof. By Theorem 3.4 we have that $\Delta + C(x)I$ generates an order preserving analytic semigroup in $X = L^q_\rho(\mathbb{R}^N)$, that we denote $S_C(t)$, which has the same fractional power spaces than $-\Delta$.

As $D \in L^q_\rho(\mathbb{R}^N)$, the linear problem

$$\begin{cases} U_t - \Delta U = C(x)U + D(x), & x \in \mathbb{R}^N,\ t > 0 \\ U(0) = |u_0| \in H^{1,q}_\rho(\mathbb{R}^N) \end{cases} \tag{4.9}$$

is well defined and has a unique solution, $U(t, |u_0|)$, which is given by the variations of constants formula

$$U(t, |u_0|) = S_C(t)|u_0| + \int_0^t S_C(t-s)D(x)ds \tag{4.10}$$

and satisfies $U(\cdot, |u_0|) \in C([0,\infty), H^{1,q}_\rho(\mathbb{R}^N)) \cap C((0,\infty), H^{2,q}_\rho(\mathbb{R}^N))$, $U(t,x) \geq 0$ for every $x \in \mathbb{R}^N$ and $t > 0$, since $D \geq 0$.

By comparison, we have

$$|u(t,u_0)| \leq U(t, |u_0|) \tag{4.11}$$

for all $t > 0$.

Now using the weighted Sobolev inclusions in Lemma 2.12, we obtain that for

$$q \leq p \begin{cases} < \infty, & \text{if } N \leq q \\ \leq \frac{qN}{N-q}, & \text{if } 1 \leq q < N \end{cases}$$

we have

$$\|u(s)\|_{L^p_{\rho^{\frac{p}{q}}}(\mathbb{R}^N)} \leq \|U(s)\|_{L^p_{\rho^{\frac{p}{q}}}(\mathbb{R}^N)} \leq C\|U(s)\|_{H^{1,q}_\rho(\mathbb{R}^N)}$$

and from (4.10)

$$\begin{aligned} \|U(t,|u_0|)\|_{H^{1,q}_\rho(\mathbb{R}^N)} &\leq \|S_C(t)|u_0|\|_{H^{1,q}_\rho(\mathbb{R}^N)} + \int_0^t \|S_C(t-s)D(x)\|_{H^{1,q}_\rho(\mathbb{R}^N)}ds. \\ &\leq Me^{\mu t}\|u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)} + M\|D\|_{L^q_\rho(\mathbb{R}^N)} \int_0^t \frac{e^{\mu(t-s)}}{(t-s)^{\frac{1}{2}}}\, ds \end{aligned}$$

for some $\mu \in \mathbb{R}$. Therefore, for all $T > 0$, if $t \in [0,T]$,

$$\|U(t,|u_0|)\|_{H^{1,q}_\rho(\mathbb{R}^N)} \leq K(T)\big(\|u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)} + \|D\|_{L^q_\rho(\mathbb{R}^N)}\big) \tag{4.12}$$

for some $K(T) > 0$.

Using now the variations of constants formula we get estimates of $u(t,u_0)$ in $H^{1,q}_\rho(\mathbb{R}^N)$. In fact, using (4.4), we get, for some $\alpha\in\mathbb{R}$

$$\begin{aligned}\|u(t,u_0)\|_{H^{1,q}_\rho(\mathbb{R}^N)} &\leq \|e^{(\Delta+m(x)I)t}u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)}+\int_0^t \|e^{(\Delta+m(x)I)(t-s)}(g+f_0^e(u(s)))\|_{H^{1,q}_\rho(\mathbb{R}^N)}\,ds\\ &\leq Me^{\alpha t}\|u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)}+M\int_0^t\frac{e^{\alpha(t-s)}}{(t-s)^{\frac12}}\|g+f_0^e(u(s))\|_{L^q_\rho(\mathbb{R}^N)}\,ds.\end{aligned}$$

Thus,

$$\|u(t)\|_{H^{1,q}_\rho(\mathbb{R}^N)}\leq Me^{\alpha t}\|u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)}+M\sup_{s\in[0,T]}\|g+f_0^e(u(s))\|_{L^q_\rho(\mathbb{R}^N)}\int_0^t\frac{e^{\alpha(t-s)}}{(t-s)^{\frac12}}\,ds$$

and then

$$\|u(t)\|_{H^{1,q}_\rho(\mathbb{R}^N)}\leq K(T)\big(\|u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)}+\sup_{s\in[0,T]}\|g+f_0^e(u(s))\|_{L^q_\rho(\mathbb{R}^N)}\big). \tag{4.13}$$

Using now (4.6)

$$\begin{aligned}\|g+f_0^e(u(s))\|^q_{L^q_\rho(\mathbb{R}^N)} &\leq C\|g\|^q_{L^q_\rho(\mathbb{R}^N)}+C(\|u(s)\|^q_{L^q_\rho(\mathbb{R}^N)}+\|u(s)\|^{qr}_{L^{qr}_{\rho^r}(\mathbb{R}^N)}),\\ &\leq C\|g\|^q_{L^q_\rho(\mathbb{R}^N)}+C(\|U(s)\|^q_{H^{1,q}_\rho(\mathbb{R}^N)}+\|U(s)\|^{qr}_{H^{1,q}_\rho(\mathbb{R}^N)}).\end{aligned} \tag{4.14}$$

Hence, by (4.12), we obtain that for $s\in[0,T]$, with $T<\infty$

$$\|g+f_0^e(u(s))\|_{L^q_\rho(\mathbb{R}^N)}\leq C(T,\|u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)},\|g\|_{L^q_\rho(\mathbb{R}^N)},\|D\|_{L^q_\rho(\mathbb{R}^N)}).$$

Plugging this into (4.13) we obtain that for all $0\leq t\leq T<\infty$ we have

$$\|u(t)\|_{H^{1,q}_\rho(\mathbb{R}^N)}\leq C(T,\|u_0\|_{H^{1,q}_\rho(\mathbb{R}^N)},\|g\|_{L^q_\rho(\mathbb{R}^N)},\|D\|_{L^q_\rho(\mathbb{R}^N)}). \tag{4.15}$$

Hence the solution is global in $H^{1,q}_\rho(\mathbb{R}^N)$.□

Now we show how an additional dissipativity condition allows us to obtain uniform bounds on the solutions, independent of the initial data.

Theorem 4.4. *Let $1<q<\infty$. Assume the nonlinear term in (4.1) satisfies conditions (4.2), (4.7) with $0\leq D\in L^q_\rho(\mathbb{R}^N)$, $C\in L^\sigma_U(\mathbb{R}^N)$, $\sigma>\frac N2$, $\sigma>q$ and that the analytic semigroup in $L^q_\rho(\mathbb{R}^N)$ generated by $\Delta+C(x)I$, with domain $H^{2,q}_\rho(\mathbb{R}^N)$ decays exponentially. Let $0\leq\phi\in H^{2,q}_\rho(\mathbb{R}^N)$ be the unique solution of the elliptic problem*

$$-\Delta\phi=C(x)\phi+D(x),\ x\in\mathbb{R}^N. \tag{4.16}$$

Then the solution of (4.1) satisfies for $q\leq p<\infty$

$$\limsup_{t\to\infty}\|u(t,u_0)\|_{L^p_{\rho^{\frac pq}}(\mathbb{R}^N)}\leq C(\|\phi\|_{H^{2,q}_\rho(\mathbb{R}^N)}) \tag{4.17}$$

and, if $q>\frac N2$,

$$\limsup_{t\to\infty}\|u(t,u_0)\|_{L^\infty_{\rho^{\frac1q}}(\mathbb{R}^N)}\leq C(\|\phi\|_{H^{2,q}_\rho(\mathbb{R}^N)}) \tag{4.18}$$

and both limits are uniform for u_0 in bounded sets of $H^{1,q}_\rho(\mathbb{R}^N)$. Moreover for each $0 \le \alpha < 1$

$$\limsup_{t\to\infty} \|u(t,u_0)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le C_\alpha(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}) \tag{4.19}$$

and the limit is uniform for u_0 in bounded sets of $H^{1,q}_\rho(\mathbb{R}^N)$.

Moreover, if $|u_0(x)| \le \phi(x)$ for all $x \in \mathbb{R}^N$ then

$$|u(t,x,u_0)| \le \phi(x), \quad \text{for all } \; x \in \mathbb{R}^N, \; t > 0.$$

Finally, if $q > \frac{N}{2}$ then, uniformly for u_0 in bounded sets of $H^{1,q}_\rho(\mathbb{R}^N)$, we have

$$\limsup_{t\to\infty} |u(t,x,u_0)| \le \phi(x) \tag{4.20}$$

where the limit is uniform on compact sets of $\mathbb{R}^N$ if $\inf_{x\in\mathbb{R}^N}\rho(x) = 0$, or uniformly in $\mathbb{R}^N$ if $\inf_{x\in\mathbb{R}^N}\rho(x) > 0$.

Proof. Observe first that $0 \le \phi \in H^{2,q}_\rho(\mathbb{R}^N)$ is well defined. Now we decompose the solution U of (4.9) as $U = v + \phi$, where ϕ is the solution of (4.16) and v satisfies the linear homogeneous equation

$$\begin{cases} v_t + (-\Delta - C(x)I)v = 0 \\ v(0) = |u_0| - \phi, \end{cases}$$

that is, $v(t) = e^{(\Delta + C(x)I)t}(|u_0| - \phi)$, and satisfies for every $0 \le \alpha \le 1$

$$\|v(t)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le \frac{Me^{-at}}{t^\alpha}[\|u_0\|_{L^q_\rho(\mathbb{R}^N)} + \|\phi\|_{L^q_\rho(\mathbb{R}^N)}] \tag{4.21}$$

for some $a > 0$. Therefore,

$$\|U(t)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le \|v(t)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} + \|\phi\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}. \tag{4.22}$$

Taking the limsup we get, for each $0 \le \alpha \le 1$

$$\limsup_{t\to\infty} \|U(t)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le \|\phi\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}, \tag{4.23}$$

and the limit is uniform for u_0 in bounded sets of $H^{1,q}_\rho(\mathbb{R}^N)$.

Using this estimate and (4.11) and the Sobolev embeddings in Lemma 2.12, we obtain (4.17) and (4.18).

Moreover, if $|u_0(x)| \le \phi(x)$ for all $x \in \mathbb{R}^N$, then $v(t,x) \le 0$ in $\mathbb{R}^N$ and $|u(t,x,u_0)| \le U(t,x,|u_0|) \le \phi(x)$ in $\mathbb{R}^N$.

Also, if $q > \frac{N}{2}$, taking $2\alpha > \frac{N}{q}$ in (4.21), we get that $v(t)$ tends to zero in $L^\infty_{\rho^{\frac{1}{q}}}(\mathbb{R}^N)$. This implies the convergence to zero on compact sets in $\mathbb{R}^N$ if $\inf_{x\in\mathbb{R}^N}\rho(x) = 0$, or uniform in $\mathbb{R}^N$ if $\inf_{x\in\mathbb{R}^N}\rho(x) > 0$. From this and (4.11), we get (4.20).

Finally, if $B \subset H^{1,q}_\rho(\mathbb{R}^N)$ is a bounded set of intial data, with $\alpha = \frac{1}{2}$ in (4.23), we get in particular

$$\limsup_{t\to\infty} \|u(t,u_0)\|_{L^p_{\rho^{\frac{p}{q}}}(\mathbb{R}^N)} \le C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}) \tag{4.24}$$

for $q \leq p \begin{cases} < \infty, & \text{if } N \leq q \\ \leq \frac{qN}{N-q}, & \text{if } 1 \leq q < N \end{cases}$, and from this, by (4.14) and (4.23)

$$\limsup_{t\to\infty} \|g + f_0^e(u)\|_{L^q_\rho(\mathbb{R}^N)} \leq C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}). \tag{4.25}$$

On the other hand, taking λ large enough such that the linear semigroup generated by $\Delta + m(x)I - \lambda I$ decays exponentially in $L^q_\rho(\mathbb{R}^N)$, we consider, instead of (4.1),

$$u_t - \Delta u - m(x)u + \lambda u = g(x) + \lambda u + f_0(x, u).$$

Using the corresponding variation of constants formula, we have

$$u(t+1, u_0) = S_\lambda(1)u(t) + \int_t^{t+1} S_\lambda(t+1-s)(g + \lambda u(s) + f_0^e(u(s)))\, ds \tag{4.26}$$

where $S_\lambda(t)$ denotes the linear semigroup generated by $\Delta + m(x)I - \lambda I$.

Taking the norm in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$ in (4.26), we get that for $0 \leq \alpha < 1$ and some $b > 0$

$$\|u(t+1, u_0)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \leq e^{-b}\|u(t)\|_{L^q_\rho(\mathbb{R}^N)} + \int_t^{t+1} \frac{e^{-b(t+1-s)}}{(t+1-s)^\alpha} \|g + \lambda u(s) + f_0^e(u(s))\|_{L^q_\rho(\mathbb{R}^N)}\, ds \tag{4.27}$$

Let $C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)})$ the largest of the constants in (4.24) and (4.25), then, taking for example $C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}) + 1$, and given a set of initial data B, bounded in $H^{1,q}_\rho(\mathbb{R}^N)$, by (4.24) and (4.25) there exists $T = T(B) > 0$ such that

$$\begin{cases} \|u(t, u_0)\|_{L^p_{\rho^{\frac{p}{q}}}(\mathbb{R}^N)} \leq C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}) + 1, \\ \|g + \lambda u(t) + f_0^e(u(t))\|_{L^q_\rho(\mathbb{R}^N)} \leq C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}) + 1 \end{cases}$$

for all $t \geq T(B)$.

Substituting this estimate in (4.27) we get that, for all $t \geq T(B)$

$$\begin{aligned} \|u(t+1, u_0)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} &\leq e^{-b}\left[C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}) + 1\right] \\ &+ \left[C(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}) + 1\right] \int_t^{t+1} \frac{e^{-b(t+1-s)}}{(t+1-s)^\alpha}\, ds. \end{aligned}$$

Since $\int_t^{t+1} \frac{e^{-b(t+1-s)}}{(t+1-s)^\alpha}\, ds = \int_0^1 \frac{e^{-b(1-s)}}{(1-s)^\alpha}\, ds = L_\alpha$, we conclude that for each $0 \leq \alpha < 1$ and for all $t \geq T(B) + 1$

$$\|u(t, u_0)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \leq C_\alpha(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)}). \tag{4.28}$$

Hence, for each $0 \leq \alpha < 1$ we get (4.19). □

Note that inequality (4.28) implies that the ball

$$B_{0,\alpha} = \{u \in H^{2\alpha,q}_\rho(\mathbb{R}^N) : \|u\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \leq C_\alpha(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)})\}$$

is an absorbing ball in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$ for $\frac{1}{2} \leq \alpha < 1$, for the nonlinear semigroup $\{T(t)\}_{t\geq 0}$, although it is not positively invariant. The next result allows us to find a bounded, absorbing and positively invariant set.

Lemma 4.5. *There exists a bounded set $\hat{B}_{0,\alpha}$ in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$ positively invariant, that is, $T(t)\hat{B}_{0,\alpha} \subset \hat{B}_{0,\alpha}$, $t \geq 0$ and absorbing, that is, for each bounded set B in $H^{1,q}_\rho(\mathbb{R}^N)$ there exists $t_0(B) > 0$ such that $T(t)B \subset \hat{B}_{0,\alpha}$, for all $t \geq t_0(B)$.*

Proof. We define

$$\hat{B}_{0,\alpha} := \cup_{t\geq 0} T(t) B_{0,\alpha}$$

which is positively invariant for $\{T(t)\}_{t\geq 0}$. In fact, if $z \in \hat{B}_{0,\alpha}$ then there exists $\tau_0 \geq 0$ such that $z \in T(\tau_0)B_{0,\alpha}$ and then $z = T(\tau_0)b_0$, with $b_0 \in B_{0,\alpha}$. Hence, we have $T(t)z = T(t)T(\tau_0)b_0 = T(t+\tau_0)b_0 \in \cup_{t\geq 0} T(t)B_{0,\alpha} = \hat{B}_{0,\alpha}$, for all $t \geq 0$.

Thus, $\hat{B}_{0,\alpha}$ satisfies the statement.□

5 Asymptotic compactness and the global attractor

In this section we prove that the nonlinear semigroup $\{T(t)\}_{t\geq 0}$ obtained above is compact when t goes to infinity, that is, we prove the asymptotic compactness, see [19].

Definition 5.1. *A semigroup $\{T(t)\}_{t\geq 0}$ in a Banach space X is said asymptotically compact in a Banach space Y if and only if for any sequence of initial data, bounded in X, $\{u_0^n\}$, and for any sequence $t_n \to +\infty$ then $\{T(t_n)u_0^n\}_{n\geq 1}$ has a converging subsequence in Y.*

We first show the asymptotic compactness of $\{T(t)\}_{t\geq 0}$ in $L^q_\rho(\mathbb{R}^N)$. Then we will use the variations of constants formula to conclude that $\{T(t)\}_{t\geq 0}$ is asymptotically compact in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$, for any $0 \leq \alpha < 1$.

Theorem 5.2. *Under the assumptions in Theorem 4.4, the nonlinear semigroup $\{T(t)\}_{t\geq 0}$ in $H^{1,q}_\rho(\mathbb{R}^N)$ defined by (4.1) is asymptotically compact in $L^q_\rho(\mathbb{R}^N)$.*

Proof. Take $\{u_0^n\}$ a bounded set of initial conditions in $H^{1,q}_\rho(\mathbb{R}^N)$ and $t_n \to \infty$. We show now that for each $\varepsilon > 0$ there exists $k = k_0(\varepsilon) > 0$, $n_0(\varepsilon)$ such that for all $k \geq k_0$, $n > n_0(\varepsilon)$

$$\int_{|x|>k} |u(t_n, u_0^n)|^q \rho(x)\, dx < \varepsilon, \tag{5.1}$$

that is, the solutions of (4.1) are asymptotically, uniformly small in the sense of $L^q_\rho(\mathbb{R}^N)$.

In fact, for $\varepsilon > 0$ fixed, using that from (4.21), $v(t,x) = v(t,x,|u_0| - \phi)$, converges exponentially to zero in $L^q_\rho(\mathbb{R}^N)$ as $t \to +\infty$, and the convergence is uniform for $u_0 \in B$, where B is a bounded set in $H^{1,q}_\rho(\mathbb{R}^N)$, we obtain that there exists $t_0(\varepsilon, B)$ such that for all $t \geq t_0(\varepsilon)$ and $u_0 \in B$

$$\|v(t)\|^q_{L^q_\rho(\mathbb{R}^N)} < \varepsilon. \tag{5.2}$$

From the integrability of $\phi \in L^q_\rho(\mathbb{R}^N)$, see Theorem 4.4, there exists $k_0(\varepsilon)$ such that for all $k \geq k_0(\varepsilon)$

$$\int_{|x|>k} |\phi(x)|^q \rho(x)\, dx \leq \varepsilon. \tag{5.3}$$

Using (5.2), (5.3), (4.11) and (4.22) with $\alpha = 0$, we get that for all $t \geq t_0 = t_0(\varepsilon, B)$, and all $k \geq k_0 = k_0(\varepsilon)$,

$$\int_{|x|>k} |u(t,u_0)|^q \rho(x)\, dx \leq C\Big(\int_{|x|>k} |v(t)|^q \rho(x)\, dx + \int_{|x|>k} |\phi(x)|^q \rho(x)\, dx \Big)$$
$$\leq \; C\Big(\int_{\mathbb{R}^N} |v(t)|^q \rho(x)\, dx + \int_{|x|>k} |\phi(x)|^q \rho(x)\, dx \Big) \leq 2\varepsilon C \tag{5.4}$$

and we get (5.1).

Denote, for $k > 0$, $\Omega_k = B(0,k)$. Then we now show that $\{u(t_n, u_0^n)_{|\Omega_k}\}_{n\geq 1}$ is a precompact set in $L^q(\Omega_k)$. In fact, $\{u(t_n, u_0^n)_{|\Omega_k}\}$ is bounded in $H^{1,q}(\Omega_k)$ and by the compactness of the inclusion $H^{1,q}(\Omega_k) \hookrightarrow L^q(\Omega_k)$ we conclude that $\{u(t_n, u_0^n)_{|\Omega_k}\}$ is precompact.

To end the proof, we show that for every $\varepsilon > 0$ there exists a finite covering in $L^q_\rho(\mathbb{R}^N)$ of the set $\{u(t_n, u_0^n)\}_{n\geq 1}$ by balls of radius not larger than $C\varepsilon^{\frac{1}{q}}$, for some positive constant C.

For this, let $\varepsilon > 0$, $k(\varepsilon)$ and $n \geq n_0(\varepsilon)$ be as in (5.1). Since $\{u(t_n, u_0^n)_{|\Omega_{k(\varepsilon)}}\}$ is precompact in $L^q(\Omega_{k(\varepsilon)})$, we have that $\{u(t_n, u_0^n)_{|\Omega_{k(\varepsilon)}}\}_{n\geq 1} \subset \cup_{i=1}^m B(w_i, \varepsilon_i)$, where $w_i \in L^q(\Omega_{k(\varepsilon)})$ and $\varepsilon_i \leq \varepsilon^{\frac{1}{q}}$.

Define $\hat{w}_i$ as the extension by 0 of w_i to $\mathbb{R}^N$, that is,

$$\hat{w}_i(x) = \begin{cases} w_i(x), & \text{if } x \in \Omega_{k(\varepsilon)} \\ 0, & \text{if, } x \notin \Omega_{k(\varepsilon)}. \end{cases}$$

We now show that $\{u(t_n, u_0^n)\}_{n\geq n_0} \subset \cup_{i=1}^m B(\hat{w}_i, C\varepsilon^{\frac{1}{q}})$, for some constants C. In fact, if $z \in \{u(t_n, u_0^n)\}_{n\geq n_0}$ then there exists j with $1 \leq j \leq m$, such that $\|z - w_j\|_{L^q(\Omega_{k(\varepsilon)})} < \varepsilon_j$. Hence,

$$\|z - \hat{w}_j\|^q_{L^q_\rho(\mathbb{R}^N)} \leq \|z\|^q_{L^q_\rho(\mathbb{R}^N\setminus\Omega_{k(\varepsilon)})} + C(\rho)\|z - \hat{w}_j\|^q_{L^q(\Omega_{k(\varepsilon)})} \leq \varepsilon + C(\rho)\varepsilon_j^q = C\varepsilon. \square$$

Now we use the variations of constants formula and the regularity of the semigroup to improve Theorem 5.2, proving the asymptotic compactness in the spaces $H^{2\alpha,q}_\rho(\mathbb{R}^N)$, with $\frac{1}{2} \leq \alpha < 1$. Before, we state a previous result.

Lemma 5.3. *The nonlinear semigroup $\{T(t)\}_{t\geq 0}$ in $H^{1,q}_\rho(\mathbb{R}^N)$, defined by (4.1) satisfies that for any bounded set B in $H^{1,q}_\rho(\mathbb{R}^N)$, for all $T > 0$ and any $0 < t \leq T < \infty$, $\frac{1}{2} \leq \alpha < 1$,*

$$\|T(t)u - T(t)v\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \leq \frac{L(T,B)}{t^\alpha}\|u - v\|_{L^q_\rho(\mathbb{R}^N)} \quad \textit{for} \quad u, v \in B.$$

Proof. Fix $\frac{1}{2} \leq \alpha < 1$. By the variations of constants formula

$$T(t)u = e^{(\Delta + m(x)I)t}u + \int_0^t e^{(\Delta+m(x)I)(t-s)}(f_0^e(T(s)u) + g)\, ds.$$

Then, if $u, v \in B$ is bounded in $H^{1,q}_\rho(\mathbb{R}^N)$, $t > 0$,

$$T(t)u - T(t)v = e^{(\Delta+m(x)I)t}(u - v) + \int_0^t e^{(\Delta+m(x)I)(t-s)}[f_0^e(T(s)u) - f_0^e(T(s)v)]\, ds.$$

Taking the norm in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$ and using the smoothing of the linear semigroup, we get

$$\|T(t)u - T(t)v\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le \frac{e^{\mu t}}{t^\alpha}\|u-v\|_{L^q_\rho(\mathbb{R}^N)} + \int_0^t \frac{e^{\mu(t-s)}}{(t-s)^\alpha}\|f^e_0(T(s)u) - f^e_0(T(s)v)\|_{L^q_\rho(\mathbb{R}^N)}\, ds$$

$$\le \frac{e^{\mu t}}{t^\alpha}\|u-v\|_{L^q_\rho(\mathbb{R}^N)} + C_0(B,T)\int_0^t \frac{e^{\mu(t-s)}}{(t-s)^\alpha}\|T(s)u - T(s)v\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}\, ds$$

for some $\mu \in \mathbb{R}$, for all $T>0$, if $0 \le t \le T < \infty$.

Note that we have used in the last inequality that $f^e_0 : H^{1,q}_\rho(\mathbb{R}^N) \to L^q_\rho(\mathbb{R}^N)$ is Lipschitz on bounded sets, see Theorem 4.1, that $\alpha \ge \frac{1}{2}$, B is bounded in $H^{1,q}_\rho(\mathbb{R}^N)$, and the estimates in $H^{1,q}_\rho(\mathbb{R}^N)$ for the solutions on bounded time intervals, see (4.15).

With all these, for u, $v \in B$, B bounded in $H^{1,q}_\rho(\mathbb{R}^N)$,

$$\|T(t)u - T(t)v\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le \frac{C_1}{t^\alpha}\|u-v\|_{L^q_\rho(\mathbb{R}^N)} + C_2\int_0^t \frac{1}{(t-s)^\alpha}\|T(s)u - T(s)v\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}\, ds$$

for $0 \le t \le T$, where $C_1 = C_1(T)$, $C_2 = C_2(T,B)$.

Using the singular Gronwall lemma, see [28] page 88, there exists $L(T,B)$ such that

$$\|T(t)u - T(t)v\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le \frac{L(T,B)}{t^\alpha}\|u-v\|_{L^q_\rho(\mathbb{R}^N)}, \quad \text{in } (0,T]. \square$$

We then have

Theorem 5.4. *Assume the hypotheses in Theorem 4.4. The nonlinear semigroup $\{T(t)\}_{t\ge 0}$ in $H^{1,q}_\rho(\mathbb{R}^N)$ defined by (4.1) is asymptotically compact in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$, for any $0 \le \alpha < 1$.*

Proof. Let $\{u^n_0\}$ be a bounded sequence of initial data in $H^{1,q}_\rho(\mathbb{R}^N)$ and $t_n \to +\infty$. Let $\tau^* > 0$ be fixed. Since $\{u^n_0\}$ is bounded in $H^{1,q}_\rho(\mathbb{R}^N)$ and $t_n - \tau^* \to +\infty$ then by the asymptotic compactness in $L^q_\rho(\mathbb{R}^N)$ there exists a subsequence of $\{u(t_n - \tau^*, u^n_0)\}_{n\ge 1}$, that we denote the same, which is of Cauchy type in $L^q_\rho(\mathbb{R}^N)$. Denote $w^n_0 := u(t_n - \tau^*, u^n_0)$. By the uniform bounds in (4.28), we have that for sufficiently large n, $\{w^n_0\}$ is bounded in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$. Hence, we now show that $u(t_n, u^n_0) = T(\tau^*)w^n_0$ is of Cauchy type in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$. For this, taking $u = w^n_0$ and $v = w^m_0$, the previous Lemma gives

$$\|u(t_n,u^n_0) - u(t_m,u^m_0)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} = \|T(\tau^*)w^n_0 - T(\tau^*)w^m_0\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le \frac{L}{(\tau^*)^\alpha}\|w^n_0 - w^m_0\|_{L^q_\rho(\mathbb{R}^N)} \tag{5.5}$$

But as $\{w^n_0\}_{n\ge 1}$ is of Cauchy type in $L^q_\rho(\mathbb{R}^N)$, from (5.5) we obtain that $u(t_n,u^n_0) = \{T(\tau^*)w^n_0\}_{n\ge 1}$ is of Cauchy type in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$. $\square$

From the previous results, the nonlinear semigroup $\{T(t)\}_{t\ge 0}$ in $H^{1,q}_\rho(\mathbb{R}^N)$ has a bounded absorbing set in $H^{1,q}_\rho(\mathbb{R}^N)$ and is asymptotically compact. Then from [19], the semigroup $\{T(t)\}_{t\ge 0}$ has a global attractor $\mathcal{A}$ in $H^{1,q}_\rho(\mathbb{R}^N)$, which satisfies

(i) $\mathcal{A}$ is compact in $H^{1,q}_\rho(\mathbb{R}^N)$

(ii) $\mathcal{A}$ is invariant, $T(t)\mathcal{A} = \mathcal{A}$, $\forall t \ge 0$.

(iii) $\mathcal{A}$ attracts each bounded set of $H^{1,q}_\rho(\mathbb{R}^N)$. Additionally, it is maximal in the class of bounded invariant sets in $H^{1,q}_\rho(\mathbb{R}^N)$.

Now we give additional regularity properties of the attractor.

Theorem 5.5. *Consider $\mathcal{A}$ the global attractor of (4.1) in $H^{1,q}_\rho(\mathbb{R}^N)$. Then*
(1) $\mathcal{A} \subset H^{2\alpha,q}_\rho(\mathbb{R}^N)$, $0 \le \alpha < 1$ and is bounded.
(2) $\mathcal{A}$ attracts bounded sets of $H^{1,q}_\rho(\mathbb{R}^N)$ in the norm of $H^{2\alpha,q}_\rho(\mathbb{R}^N)$

Proof. (1) By (4.19), for each $0 \le \alpha < 1$

$$\limsup_{t\to\infty} \|u(t,u_0)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le C_\alpha(\|\phi\|_{H^{1,q}_\rho(\mathbb{R}^N)}, \|g\|_{L^q_\rho(\mathbb{R}^N)})$$

uniformly for $u_0 \in B$, a bounded set of $H^{1,q}_\rho(\mathbb{R}^N)$. Now take $B = \mathcal{A}$ and then there exists $T^* = T^*(\mathcal{A})$ such that for every $t \ge T^*$ we have $\|T(t,u_0)\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le C_\alpha + 1$ with $u_0 \in \mathcal{A}$. Using the invariance of the attractor we get $\|\mathcal{A}\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \le C_\alpha + 1$. Thus, $\mathcal{A} \subset H^{2\alpha,q}_\rho(\mathbb{R}^N)$, for each $0 \le \alpha < 1$ and is bounded.

(2) We now that $\mathcal{A}$ attracts bounded sets of $H^{1,q}_\rho(\mathbb{R}^N)$ in the norm of $H^{1,q}_\rho(\mathbb{R}^N)$, that is, if B is bounded in $H^{1,q}_\rho(\mathbb{R}^N)$ then $dist_{H^{1,q}_\rho(\mathbb{R}^N)}(T(t)B, \mathcal{A}) \to 0$, as $t \to \infty$. That is for each $\{b_n\} \subset B$ and $\{t_n\} \to \infty$, there exists a sequence $\{a_n\} \subset \mathcal{A}$, such that, as $n \to \infty$,

$$\|T(t_n)b_n - a_n\|_{H^{1,q}_\rho(\mathbb{R}^N)} \to 0. \tag{5.6}$$

Let us now show that if B is bounded in $H^{1,q}_\rho(\mathbb{R}^N)$ then $dist_{H^{2\alpha,q}_\rho(\mathbb{R}^N)}(T(t)B, \mathcal{A}) \to 0$ as $t \to \infty$. We argue by contradiction. Assume there exists $\varepsilon > 0$, a sequence $\{t_n\}_{n\ge1}$, $t_n \to \infty$ such that for each $\{a_n\}_{n\ge1} \subset \mathcal{A}$ and $\{b_n\}_{n\ge1} \subset B$, we have

$$\|T(t_n)b_n - a_n\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} > \varepsilon. \tag{5.7}$$

Let ε and $\{t_n\}$ as in (5.7) and let t^* be fixed. By (5.6) applied to $t_n - \tau^* \to +\infty$, there exists $\{a_n\}_{n\ge1} \subset \mathcal{A}$, and a sequence $\{b_n\}_{n\ge1} \subset B$ such that

$$\|T(t_n - t^*)b_n - a_n\|_{H^{1,q}_\rho(\mathbb{R}^N)} \to 0.$$

Denoting, $\hat{b}_n := T(t_n - t^*)b_n$ and $\hat{\hat{a}}_n = T(t^*)a_n \in \mathcal{A}$, by Lemma 5.3 we have that there exists $L > 0$ such that for every $n \in \mathbb{N}$

$$\begin{aligned}\|T(t_n)b_n - \hat{\hat{a}}_n\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} = \|T(t^*)\hat{b}_n \quad &- \quad T(t^*)a_n\|_{H^{2\alpha,q}_\rho(\mathbb{R}^N)} \\ &\le \quad L(T)\frac{1}{(t^*)^\alpha}\|\hat{b}_n - a_n\|_{L^q_\rho(\mathbb{R}^N)} \overset{n\to\infty}{\to} 0\end{aligned}$$

which contradicts (5.7). □

With these we can prove

Theorem 5.6. *(i) The nonlinear semigroup $\{T(t)\}_{t\ge0}$ constructed above is well defined in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$ for $\frac{1}{2} \le \alpha < 1$.*
(ii) There exists a bounded absorbing set in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$.
(iii) $\{T(t)\}_{t\ge0}$ is asymptotically compact in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$.
Therefore, $\{T(t)\}_{t\ge0}$ has a global attractor in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$, $\frac{1}{2} \le \alpha < 1$ which is independent of α and coincides with the attractor in $H^{1,q}_\rho(\mathbb{R}^N)$.

Proof. Part (*i*) is immediate. Part (*ii*) is given by Lemma 4.5. For (*iii*) see Theorem 5.4. Last part is obtained from [19], since the attractor in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$ denoted $\mathcal{A}^\alpha$, exists. Since $\mathcal{A}$ and $\mathcal{A}^\alpha$ are bounded and invariant in $H^{2\alpha,q}_\rho(\mathbb{R}^N)$ for $\alpha \geq \frac{1}{2}$ then, by the maximality of $\mathcal{A}^\alpha$ we get $\mathcal{A} \subset \mathcal{A}^\alpha$. Analogously, $\mathcal{A}$, $\mathcal{A}^\alpha$ are bounded and invariant in $H^{1,q}_\rho(\mathbb{R}^N)$ and then the maximality of $\mathcal{A}$ in $H^{1,q}_\rho(\mathbb{R}^N)$ we get $\mathcal{A}^\alpha \subset \mathcal{A}$. Hence, $\mathcal{A} = \mathcal{A}^\alpha$.□

Finally, we now prove that the attractor constructed above has extremal equilibria, see [25, 26, 13].

Theorem 5.7. *Under the assumption of Theorem 4.4, there exist two extremal ordered equilibria* $\varphi_m \leq \varphi_M$*,* $\varphi_m, \varphi_M \in H^{2,q}_\rho(\mathbb{R}^N)$*, such that*

$$\varphi_m(x) \leq \liminf_{t\to\infty} u(t, x; u_0) \leq \limsup_{t\to\infty} u(t, x; u_0) \leq \varphi_M(x) \tag{5.8}$$

for $x \in \mathbb{R}^N$ *and uniformly on bounded sets of initial data in* $H^{1,q}_\rho(\mathbb{R}^N)$*. The attractor of (4.1) satisfies*

$$\mathcal{A} \subset [\varphi_m, \varphi_M], \quad \varphi_m, \varphi_M \in \mathcal{A}.$$

Moreover, φ_M *is globally asymptotically stable form above in* $H^{1,q}_\rho(\mathbb{R}^N)$ *and* φ_m *is so from below. In particular, any equilibrium (4.1) stays between the two extremal equilibria.*

If we assume additionaly that $q > \frac{N}{2}$*, then (5.8) holds uniformly in compact sets of* $\mathbb{R}^N$ *if* $\inf_{x\in\mathbb{R}^N} \rho(x) = 0$*, or uniformly in* $\mathbb{R}^N$ *if* $\inf_{x\in\mathbb{R}^N} \rho(x) > 0$*.*

Proof. Observe that $0 \leq \phi$ in (4.16) is a supersolution for (4.1) since

$$-\Delta\phi = C(x)\phi + D(x) \geq f(x, \phi).$$

Hence, the solution of (4.1) with initial data ϕ satisfies $T(t)\phi \leq \phi$ and therefore it is decreasing in time. As the semigroup is asymptotically compact, the ω–limit set of this trajectory is a unique equilibrium point, that is

$$\lim_{t\to\infty} T(t)\phi = \varphi_M \quad \text{in} \quad H^{1,q}_\rho(\mathbb{R}^N).$$

Let $B \in H^{1,q}_\rho(\mathbb{R}^N)$ a bounded set of intial data. Then for every $u_0 \in B$ we have (4.11), that is

$$|u(t, u_0)| \leq U(t, |u_0|)$$

and $U(t) = \phi + v(t)$, where $v(t) \to 0$ in $H^{1,q}_\rho(\mathbb{R}^N)$, as $t \to \infty$, uniformly in $u_0 \in B$, see (4.21).

In particular, for each $t > 0$, $u(t, x, u_0) \leq U(t, x, |u_0|)$ and using the nonlinear semigroup at time $s > 0$ we have

$$u(t + s, x, u_0) = T(s)u(t, x, u_0) \leq T(s)U(t, x, |u_0|).$$

Passing to the limit as $t \to \infty$ and using the continuity of the nonlinear semigroup in $H^{1,q}_\rho(\mathbb{R}^N)$ we get

$$\limsup_{t\to\infty} u(t, x; u_0) \leq T(s)\phi(x)$$

for $x \in \mathbb{R}^N$. Taking now the limit $s \to \infty$ we get the last inequality in (5.8), uniformly in $u_0 \in B$.

Arguing with $-\phi$ and using $u(t,x,u_0) \geq U(t,x,-|u_0|)$, we get the minimal equilibrium φ_m and we obtain (5.8).

Finally, if $q > \frac{N}{2}$ observe that by Theorems 5.5 and 5.6 we can repeat the arguments above, using now the convergence and the conibuity of the semigroup in $H^{2\alpha,q}_{\rho}(\mathbb{R}^N)$ with $\alpha < 1$. Also, by the Sobolev inclusions in Lemma 2.12 this convergence implies convergence in $L^{\infty}_{\rho^{\frac{1}{q}}}(\mathbb{R}^N)$. This convergence, in turn, implies uniform convergence in compact sets of $\mathbb{R}^N$ if $\inf_{x\in\mathbb{R}^N} \rho(x) = 0$, or uniformly in $\mathbb{R}^N$ if $\inf_{x\in\mathbb{R}^N} \rho(x) > 0$.□

Now we test our results with the important model example of logistic equations. In fact we have,

Proposition 5.8. *Suppose that*

$$f(x,s) = m(x)s - n(x)|s|^{r-1}s, \quad x \in \mathbb{R}^N, \ s \in \mathbb{R},$$

where $r > 1$,

$$m \in L^{\sigma}_{U}(\mathbb{R}^N), \quad \sigma > \frac{N}{2}$$

and there exists

$$0 < \rho_0 \in R_{\rho_1,\rho_2}$$

such that

$$0 \leq n(x) \leq \rho_0(x) \quad \textit{for } x \in \mathbb{R}^N.$$

Moreover, assume

$$1 \leq r \begin{cases} < \infty, & \textit{if } N \leq \sigma \\ \leq \frac{N}{N-\sigma}, & \textit{if } \frac{N}{2} < \sigma < N. \end{cases}$$

Then for any q *such that*

$$q_0(r) := \max\{\frac{N}{r'}, 1\} < q < \sigma,$$

the problem (4.1) is well posed in $H^{1,q}_{\rho}(\mathbb{R}^N)$ *where*

$$\rho(x) = \rho_0(x)^{\frac{q}{r-1}}.$$

Proof. Note that with the notations in (4.2), we have $f_0(x,s) = -n(x)|s|^{r-1}s$ and then $|f_0(x,s)| \leq |s|^r \rho(x)^{\frac{r-1}{q}}$ with $\rho(x) = \rho_0(x)^{\frac{q}{r-1}}$. Note that by Lemma 2.4 we have $\rho \in R_{C,C}$ for some constant C. Also (4.3) is satisfied as soon as $q > \frac{N}{r'}$. Finally the other restrictions in Theorem 4.1, namely $1 < q < \sigma$ are satisfied due to the restrictions on r and σ in the statement.□

Concerning the global existence and the asymptotic behavior note that we just need to prove that (4.7) holds with $0 \leq D \in L^q_{\rho}(\mathbb{R}^N)$, $C \in L^{\sigma}_{U}(\mathbb{R}^N)$, $\sigma > \frac{N}{2}$, $\sigma > q$ and such that the analytic semigroup in $L^q_{\rho}(\mathbb{R}^N)$ generated by $\Delta + C(x)I$, with domain $H^{2,q}_{\rho}(\mathbb{R}^N)$ decays exponentially in $L^q_{\rho}(\mathbb{R}^N)$. Hence we have

Proposition 5.9. *With the notations in Proposition 5.8, consider*

$$q_0(r) := \max\{\frac{N}{r'}, 1\} < q < \sigma,$$

and

$$\rho(x) = \rho_0(x)^{\frac{q}{r-1}}.$$

Assume there exists a decomposition

$$m(x) = m_1(x) + m_2(x), \quad m_2 \geq 0$$

such that $m_1 \in L^\sigma_U(\mathbb{R}^N)$*,* $\sigma > \frac{N}{2}$*, the analytic semigroup generated by* $\Delta + m_1(x)I$*, decays exponentially in* $L^q_\rho(\mathbb{R}^N)$*.*

Assume moreover that

$$\frac{m_2}{n^{1/r}} \in L^{qr'}_\rho(\mathbb{R}^N).$$

Then, Theorems 4.4, 5.2, 5.4, 5.5, 5.6 and 5.7 apply.

Proof. Note that, using Young's inequality, we have

$$f(x, s)s \leq m_1(x)s^2 + C\Big(\frac{m_2(x)}{n^{1/r}(x)}\Big)^{r'}|s|,$$

for some constant $C > 0$. Therefore (4.7) is satisfied with $C(x) = m_1(x)$ and $D(x)$ a multiple of $\left(\frac{m_2(x)}{n^{1/r}(x)}\right)^{r'}$. Thus, with q and $\rho(x)$ as in Proposition 5.8, we have that the conditions in the statement guarantee that $D \in L^q_\rho(\mathbb{R}^N)$.□

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