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A note on the Liouville method applied to elliptic eventually degenerate fully nonlinear equations governed by the Pucci operators and the Keller–Osserman condition

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A note on the Liouville method applied to elliptic eventually degenerate fully nonlinear equations governed by the Pucci operators and the Keller–Osserman condition

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To Ventura Reyes Prósper, many years later

Abstract

This paper is devoted to the study of the viscosity solutions of

$$\mathbb{F}(\mathrm{D}^2 u, u, x) + f = 0$$

in the whole space $\mathbb{R}^{\mathrm{N}}$, under suitable structural assumptions on $\mathbb{F}$ involving the Pucci extremal operators for the leading part and the Keller–Osserman condition on the zeroth order term. By means a kind of Liouville method we prove that uniqueness of solutions holds. Therefore a unique growth at infinity is compatible with the structure of the equations. Furthermore the knowledge of these growth at infinity of solutions is not required a priori. Due to the presence of a strong absorption term, the Liouville Theorem proved is independent on the dimension.

1 Introduction. The main contributions

Essentially, we focuss the attention on the uniqueness of solutions on elliptic PDE's equations in the whole space $\mathbb{R}^{\mathrm{N}}$ without growth restrictions. Usually this property involves suitable versions of the well known Liouville Theorem. From the prime paper by H. Brezis (see [3]), where uniqueness of *very weak* solutions of the semilinear equation

$$-\Delta u + |u|^{\mathrm{m}-1} u = f \quad \text{in } \mathscr{D}'(\mathbb{R}^{\mathrm{N}}), \quad u \in \mathrm{L}^1_{loc}(\mathbb{R}^{\mathrm{N}}), \quad \mathrm{m} > 1, \tag{1}$$

other authors have studied the topic on several equations with adequate structures on divergence form (see for instance [2]).

As in [8], or as in [10], we are interested in the solutions of the fully nonlinear equation

$$\mathbb{F}(\mathrm{D}^2 u, u, x) + f = 0 \quad \text{in } \mathbb{R}^{\mathrm{N}},\ \mathrm{N} \geq 1, \tag{2}$$

where $\mathbb{F} \in \mathscr{C}(\mathscr{S}^{\mathrm{N}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}})$, $\mathbb{F} \not\equiv 0$, satisfies $\mathbb{F}(0,0,x) = 0$ as well as the structural conditions

$$\mathscr{P}^-_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}) \leq \mathbb{F}(\mathrm{X},t,x) - \mathbb{F}(\mathrm{Y},t,x) \leq \mathscr{P}^+_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}), \quad \mathrm{X},\mathrm{Y} \in \mathscr{S}^{\mathrm{N}},\ t \in \mathbb{R},\ x \in \mathbb{R}^{\mathrm{N}}, \tag{3}$$

and

$$\mathbb{F}(\mathrm{X},s,x) - \mathbb{F}(\mathrm{X},t,x) \geq a_0(x)\beta(t-s), \quad \mathrm{X} \in \mathscr{S}^{\mathrm{N}},\ t,s \in \mathbb{R},\ t \geq s,\ x \in \mathbb{R}^{\mathrm{N}}, \tag{4}$$

for some continuous and increasing function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$, $\beta(0) = 0$ and some function $a_0 \in \mathscr{C}(\mathbb{R}^{\mathrm{N}})$ satisfying the coercivity assumption

$$a_0(x) \geq \alpha > 0, \quad x \in \mathbb{R}^{\mathrm{N}}. \tag{5}$$

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By $\mathscr{S}^{\mathrm{N}}$, respectively $\mathscr{S}^{\mathrm{N}}_{+}$, one denotes the set of the real $\mathrm{N}\times\mathrm{N}$ symmetric matrices, respectively the set of the real $\mathrm{N}\times\mathrm{N}$ symmetric and nonnegative definite matrices. Moreover $0\le\lambda\le\Lambda,\ \Lambda>0$ are two given constants for which one defines

$$\begin{cases} \mathscr{P}^{-}_{\lambda,\Lambda}(\mathrm{X}) \doteq \lambda \sum\limits_{e_i(\mathrm{X})>0} e_i(\mathrm{X}) + \Lambda \sum\limits_{e_i(\mathrm{X})<0} e_i(\mathrm{X}), \quad \mathrm{X}\in\mathscr{S}^{\mathrm{N}}, \\ \mathscr{P}^{+}_{\lambda,\Lambda}(\mathrm{X}) \doteq \Lambda \sum\limits_{e_i(\mathrm{X})>0} e_i(\mathrm{X}) + \lambda \sum\limits_{e_i(\mathrm{X})<0} e_i(\mathrm{X}), \quad \mathrm{X}\in\mathscr{S}^{\mathrm{N}}, \end{cases} \tag{6}$$

where $e_i(\mathrm{X})$ are the eigenvalues of X. Since one proves

$$\begin{cases} \mathscr{P}^{-}_{\lambda,\Lambda}(\mathrm{X}) = \inf\limits_{\mathrm{Z}\in\mathscr{A}_{\lambda,\Lambda}} \textbf{trace}\,\mathrm{ZX}, \quad \mathrm{X}\in\mathscr{S}^{\mathrm{N}}, \\ \mathscr{P}^{+}_{\lambda,\Lambda}(\mathrm{X}) = \sup\limits_{\mathrm{Z}\in\mathscr{A}_{\lambda,\Lambda}} \textbf{trace}\,\mathrm{ZX}, \quad \mathrm{X}\in\mathscr{S}^{\mathrm{N}}, \end{cases}$$

for

$$\mathscr{A}_{\lambda,\Lambda} \doteq \{\mathrm{Z}\in\mathscr{S}^{\mathrm{N}}:\ \lambda|\xi|^2 \le \langle \mathrm{Z}\xi,\xi\rangle \le \Lambda|\xi|^2,\ \xi\in\mathbb{R}^{\mathrm{N}}\},$$

one says that $\mathscr{P}^{-}_{\lambda,\Lambda}, \mathscr{P}^{+}_{\lambda,\Lambda}$ are the Pucci extremal operators. We send to [4] for some properties of these operators. In particular, one proves the inequalities

$$\mathscr{P}^{-}_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}) \le \mathscr{P}^{\pm}_{\lambda,\Lambda}(\mathrm{X}) - \mathscr{P}^{\pm}_{\lambda,\Lambda}(\mathrm{Y}) \le \mathscr{P}^{+}_{\lambda,\Lambda}(\mathrm{X}-\mathrm{Y}), \quad \mathrm{X},\mathrm{Y}\in\mathscr{S}^{\mathrm{N}}.$$

In order to simplify the exposition we omit in this paper the eventual dependence on $\mathrm{D}u$ (see [8]). We note that the structure (3), (4) and (5) is more general on the dependence on D^2u and u that those considered in [8] or [10]. In (18) below we give an example which illustrates our contributions.

Since (2) is a non divergence equation the viscosity solutions theory is applicable. More precisely, in what follows we argue with semi–continuous viscosity solutions, thus with locally bounded functions u verifying

$$\begin{cases} \mathbb{F}(\mathrm{D}^2u^*, u^*, x) + f \ge 0, \\ \mathbb{F}(\mathrm{D}^2v_*, v_*, x) + g \le 0, \end{cases} \tag{7}$$

in an open set $\Omega\subseteq\mathbb{R}^{\mathrm{N}}$, in the viscosity sense (see [5]), for the upper and lower semi–continuous envelopes

$$\begin{cases} u^*(x) \doteq \lim\limits_{r\to 0}\sup\ \{u(y):\ |x-y|\le r\}, \\ u_*(x) \doteq \lim\limits_{r\to 0}\inf\ \{u(y):\ |x-y|\le r\}. \end{cases}$$

Certainly, one has

$$u_*(x) \le u(x) \le u^*(x).$$

In fact, we will prove in Theorem 2 below that under suitable assumptions the equation (2) only admits continuous viscosity solutions whenever $f\in\mathscr{C}(\mathbb{R}^{\mathrm{N}})$.

From now on we drop the term *viscosity* which is an artifact of the origin of this theory motivated by the consistency of the notion with the method of *vanishing viscosity*, mainly for first order equations. In Section 3 we give an existence result of a continuous solution of (2) (see Theorem 4). Since $\mathscr{P}^{-}_{0,\Lambda}(\mathrm{X})=0$ for $\mathrm{X}\in\mathscr{S}^{\mathrm{N}}_{+}$, whenever $\lambda=0$ condition (3) says that (2) is an elliptic degenerate fully nonlinear equation and therefore we can not expect $\mathscr{C}^2$–solutions. For the non degenerate case, $\lambda>0$, under additional assumptions on $\mathbb{F}$, we may prove, as in [4], that u is a $\mathscr{C}^2$–solution. We recall that $\mathscr{P}^{-}_{\lambda,\Lambda}(\mathrm{X})$ is concave and $\mathscr{P}^{+}_{\lambda,\Lambda}(\mathrm{X})$ is convex. Here we only prove uniform continuity of solutions (see Corollary 1).

The construction of the solution obtained in Theorem 4 (see (26)) leads to a growth at the infinity. Our main goal in the paper, as in [8], is to prove that the growth does not depend on the construction nor can be arbitrary: it only depends on the structure of the equation and consequently a unique growth at the infinity of the solutions is admissible and whose knowledge is not required a priori. It is derived from a result on the uniqueness of the solutions of (2). Certainly, our contributions extend the relative ones of [8] or [10]. Here the elliptic degenerate case ($\lambda=0$) and more general zeroth terms are considered (see Remark 2).

Essentially the main result of [3], for (1), was obtained by constructing a precise test function and by using the Kato's inequality. Due to (2) has not variational structure no Brezis test function, as in [3], is available. Therefore, we follows the same program of [8] based on a kind of Liouville method. So, let u and v be solutions of (7) in an open set $\Omega \subseteq \mathbb{R}^{\mathrm{N}}$, for $f,g \in \mathscr{C}(\Omega)$. Since inequality

$$(v_* + \mathrm{W})(x) \geq (v_* + \mathrm{W})(x_0) + \langle p, x - x_0\rangle + \frac{1}{2}\langle \mathrm{Y}(x - x_0), x - x_0\rangle + o\big(|x - x_0|^2\big)$$

implies

$$v_*(x) \geq v_*(x_0) + \langle p - \mathrm{DW}(x_0), x - x_0\rangle + \frac{1}{2}\langle (\mathrm{Y} - \mathrm{D}^2\mathrm{W}(x_0)(x - x_0), x - x_0\rangle + o\big(|x - x_0|^2\big)$$

for any $\mathscr{C}^2$ nonnegative function, W, the structural assumptions (3) and (4) lead to

$$\begin{aligned}\mathbb{F}\big(\mathrm{Y}, (v_* + \mathrm{W})(x_0), x_0\big) \leq\ & \mathscr{P}^+_{\lambda,\Lambda}\big(\mathrm{D}^2\mathrm{W}(x_0)\big) + \mathbb{F}\big(\mathrm{Y} - \mathrm{D}^2\mathrm{W}(x_0), (v_* + \mathrm{W})(x_0), x_0\big)\\ \leq\ & \mathscr{P}^+_{\lambda,\Lambda}\big(\mathrm{D}^2\mathrm{W}(x_0)\big) - a_0(x_0)\beta\big(\mathrm{W}(x_0)\big)\\ & + \mathbb{F}\big(\mathrm{Y} - \mathrm{D}^2\mathrm{W}(x_0), v_*(x_0), x_0\big),\end{aligned}$$

whence the definition of supersolution gives

$$\begin{cases}\mathbb{F}(\mathrm{D}^2u^*, u^*, x) + f \geq 0,\\ \mathbb{F}\big(\mathrm{D}^2(v_* + \mathrm{W}), v_* + \mathrm{W}, x\big) + g - \mathscr{P}^+_{\lambda,\Lambda}\big(\mathrm{D}^2\mathrm{W}\big) + a_0\beta(\mathrm{W}) \leq 0,\end{cases} \tag{8}$$

where the second order differential term $\mathscr{P}^+_{\lambda,\Lambda}\big(\mathrm{D}^2\mathrm{W}\big)$ is uniformly elliptic even if $\lambda = 0$. The main idea is to prove a Liouville Theorem on $u^* - v_*$ whence suitable consequences are deduced. Based on universal interior bounds on cubes as $\mathbf{Q}_{\mathrm{R}}(x_0) \doteq \{x \in \mathbb{R}^{\mathrm{N}} :\ \|x - x_0\|_\infty < \mathrm{R}\}$, $\|x - x_0\|_\infty \doteq \max\limits_{1\leq i\leq \mathrm{N}} |x_i - x_{0,i}|$, the Liouville Theorem is proved assuming the condition

$$\int^\infty \frac{ds}{\sqrt{\mathbb{B}(s)}} < \infty, \quad \mathbb{B}' = \beta, \tag{9}$$

due to J.B Keller and R.Osserman ([14], [15]). So, we consider the decreasing and nonnegative function

$$\Psi(t) \doteq \int_t^\infty \frac{ds}{\sqrt{\int_0^s \beta(\tau)d\tau}}, \quad t > 0.$$

Clearly, $\Psi(t)$ is well defined by (9).

Theorem 1 (Local universal bound) *Let us assume* (9). *Then for three positive constants* Λ, α *and* N, *the decreasing function*

$$\mathrm{R}_k \doteq \sqrt{\frac{\Lambda\mathrm{N}}{\alpha}\frac{\Psi(k)}{c(\mathrm{A}_\infty)}}, \quad k \in \mathbb{R}_+, \tag{10}$$

verifies

$$\lim_{k\to+\infty} \mathrm{R}_k = 0 \quad and \quad \lim_{k\to 0} \mathrm{R}_k \doteq \mathrm{R}_0 \leq \infty, \tag{11}$$

where $c(\mathrm{A}_\infty)$ *is the maximum value of the function*

$$c(\mathrm{A}) \doteq \frac{1}{\sqrt{2\left(\dfrac{1}{\mathrm{A}} + \dfrac{1}{(1-\mathrm{A})\sqrt{\mathrm{A}}}\right)}}, \quad 0 < \mathrm{A} < 1. \tag{12}$$

Moreover, one has the inequality

$$\begin{aligned}(u^* - v_*)(x) \leq\ & \frac{1}{\mathrm{A}_\infty}\sum_{i=1}^{\mathrm{N}} \Psi^{-1}\left(c(\mathrm{A}_\infty)\sqrt{\frac{\alpha}{\mathrm{N}\Lambda}}\frac{\mathrm{R}_k^2 - |x_i - x_{0,i}|^2}{\mathrm{R}_k}\right)\\ & + \beta^{-1}\left(\sup\nolimits_{\mathbf{Q}_{\mathrm{R}_k}(x_0)} \frac{(f-g)_+}{\alpha}\right),\end{aligned} \tag{13}$$

for $x \in \mathbf{Q}_{\mathrm{R}_k}(x_0) \subset \Omega$*, whenever u and v are solutions of* (7) *in* Ω*, for* $f, g \in \mathscr{C}(\Omega)$*, provided* (3), (4) *and* (5). *In particular,* $\mathbf{Q}_{\mathrm{R}_k}(x) \subset \Omega$ *implies the inequality*

$$(u^* - v_*)(x) \leq \frac{\mathrm{N}}{\mathrm{A}_\infty} k + \beta^{-1}\left(\sup_{\mathbf{Q}_{\mathrm{R}_k}(x)} \frac{(f-g)_+}{\alpha}\right). \tag{14}$$

First of all, it is clear that

$$\lim_{\mathrm{A}\to 0} c(\mathrm{A}) = \lim_{\mathrm{A}\to 1} c(\mathrm{A}) = 0,$$

consequently there exists $\mathrm{A}_\infty \in]0,1[$ where $c(\mathrm{A})$ attains their maximum value, *i.e.*

$$c(\mathrm{A}) \leq c(\mathrm{A}_\infty) < \frac{\sqrt[4]{3}}{3}, \quad 0 < \mathrm{A}_\infty < 1.$$

The proof of Theorem 1 is given in Section 2. It is based on to show that

$$\mathrm{W}_{\mathrm{R}_k}(x) \doteq \frac{1}{\mathrm{A}_\infty} \sum_{i=1}^{\mathrm{N}} \Psi_k^{-1}\left(c(\mathrm{A}_\infty)\sqrt{\frac{\alpha}{\mathrm{N}\Lambda}} \frac{\mathrm{R}_k^2 - |x_i - x_{0,i}|^2}{\mathrm{R}_k}\right) \tag{15}$$

is a classical $\mathscr{C}^2$ solution of

$$-\mathscr{P}^+_{\lambda,\Lambda}(\mathrm{D}^2\mathrm{W}_{\mathrm{R}_k}) + a_0\beta\left(\mathrm{W}_{\mathrm{R}_k}\right) \geq 0 \quad \text{in } \mathbf{Q}_{\mathrm{R}_k}(x_0). \tag{16}$$

Therefore (8) becomes

$$\begin{cases} \mathbb{F}(\mathrm{D}^2u^*, u^*, x) + f \geq 0 & \text{in } \mathbf{Q}_{\mathrm{R}_k}(x_0), \\ \mathbb{F}\left(\mathrm{D}^2(v_* + \mathrm{W}_{\mathrm{R}_k}), v_* + \mathrm{W}_{\mathrm{R}_k}, x\right) + g \leq 0 & \text{in } \mathbf{Q}_{\mathrm{R}_k}(x_0). \end{cases} \tag{17}$$

So that, the result is derived by using the behaviour

$$\mathrm{W}_{\mathrm{R}_k}(x) = \infty, \quad x \in \partial\mathbf{Q}_{\mathrm{R}_k}(x_0)$$

(see the next section for details).

Remark 1 From [15] universal bounds as (13), with $v = g \equiv 0$, are well known on balls, mainly for the semilinear equation

$$-\Delta u + |u|^{m-1}u = 0, \quad m > 1.$$

In some sense, similar bounds for

$$-\Delta u + \beta(u) = 0,$$

assuming (9), can be derived from [14] but they are more restrictive due to it requires $\Psi(0) = \infty$ (see the comments of Section 2). As in [7] or [16], other extensions are possible. Here by simplicity the universal bounds are obtained on cubes (see the comments of the next section). We note that in [10, Lemma 2.4] an interior estimate is obtained by means of the Alexandroff–Bakelman–Pucci inequality, consequently it requires the uniform ellipticity of $\mathbb{F}$, thus $\lambda > 0$. □

Remark 2 Simple choices of β verifying (9) are the power–like absorptions as $\beta_m(t) = t^m,\ m > 1$, for which

$$\Psi_m^{-1}(\zeta) = \left(\frac{m-1}{2\sqrt{m+1}}\zeta\right)^{-\frac{2}{m-1}}$$

and

$$\begin{aligned}(u^* - v_*)(x) \leq \quad & \frac{1}{\mathrm{A}_\infty}\sum_{i=1}^{\mathrm{N}}\left(\frac{c(\mathrm{A}_\infty)(m-1)}{2}\sqrt{\frac{\alpha}{\mathrm{N}\Lambda(m+1)}}\frac{\mathrm{R}_k^2 - |x_i - x_{0,i}|^2}{\mathrm{R}_k}\right)^{-\frac{2}{m-1}} \\ & + \left(\sup_{\mathbf{Q}_{\mathrm{R}_k}(x_0)} \frac{(f-g)_+}{\alpha}\right)^{\frac{1}{m}}\end{aligned}$$

where

$$R_k = \sqrt{\frac{\Lambda N(m+1)}{\alpha}\frac{2k^{-\frac{m-1}{2}}}{c(A_\infty)(m-1)}}, \quad k>0$$

and $R_0 = \infty$ (see (13)). We note that by the classical inequality

$$|t+s|^{m-1}(t+s) - |t|^{m-1}t \geq 2^{1-m}s^m, \quad t \in \mathbb{R},\ s \in \mathbb{R}_+, \quad m \geq 1,$$

the illustrative equation

$$\mathscr{P}^{\pm}_{\lambda,\Lambda}(D^2u) - a_0|u|^{m-1}u + f = 0 \tag{18}$$

is included in the cases where our contributions apply. For $a_0 \equiv 1$ it was studied in [8] and [10]. Whenever $\lambda = \Lambda = 1$ and $a_0 \equiv 1$ equation (18) becomes (1). □

An important consequence follows if Ω is the whole space $\mathbb{R}^N$. Then since $\mathbf{Q}_R(x) \subset \mathbb{R}^N$, if $0 < R < R_0$, the properties (11) imply

$$(u^* - v_*)(x) \leq \frac{N}{A_\infty}k + \beta^{-1}\left(\sup_{\mathbb{R}^N}\frac{(f-g)_+}{\alpha}\right) \quad \text{for all } k>0.$$

So, one concludes the main contribution in the paper

Theorem 2 (Comparison of solutions in the whole space) *Suppose the structural assumptions* (3), (4) *and* (5)*, as well as the Keller–Osserman condition* (9)*. Let u and v be solutions of* (7) *in* $\mathbb{R}^N$*, for* $f,g \in \mathscr{C}(\mathbb{R}^N)$ *then*

$$\|(u^* - v_*)_+\|_\infty \leq \beta^{-1}\left(\frac{\|(f-g)_+\|_\infty}{\alpha}\right). \tag{19}$$

Consequently, if u is a solution of (2) *one deduces*

$$u^* \leq u_* \quad \textit{in } \mathbb{R}^N,$$

thus, the equation (2) *only admits continuous solutions. Moreover, one has the continuous dependence on the data*

$$\|u-v\|_\infty \leq \beta^{-1}\left(\frac{\|f-g\|_\infty}{\alpha}\right). \tag{20}$$

for solutions of (2) *relatives to* $f,\ g \in \mathscr{C}(\mathbb{R}^N)$*. In particular, the equation*

$$\mathbb{F}(D^2u,u,x) + f = 0 \quad \textit{in } \mathbb{R}^N$$

has at the most a solution. So, $f \equiv 0$ *implies* $u \equiv 0$ (Liouville Theorem). □

Remark 3 Some other versions of the Liouville Theorem are well known for the Pucci extremal operators, mainly depending on the dimension N (see, for instance, [6] or [13]) Here, if $v = g \equiv 0$, (19) becomes the estimate

$$u(x) \leq u^*(x) \leq \beta^{-1}\left(\frac{\|f_+\|_\infty}{\alpha}\right), \quad x \in \mathbb{R}^N,$$

and then $f \leq 0$ implies $u \leq 0$. We note that the presence of a strong absorption term enables us to derive this version of the Liouville Theorem independent on the dimension. See also [1] were a comparison principle in the whole space is obtained. □

Let u a continuous solution of (2) and $y \in \mathbb{R}^N$. Then if $\mathbb{F}(X,r,x) \equiv \mathbb{F}(X,r)$, $a_0(x) \equiv \alpha$, the function $v(\cdot) = u(\cdot + y)$ solves

$$\mathbb{F}(D^2v,v) + g = 0 \quad \text{in } \mathbb{R}^N$$

where $g(\cdot) = f(\cdot + y)$. So, from the inequality (20) one deduces

$$|u(x+y) - u(x)| \leq \beta^{-1}\left(\sup_{z\in\mathbb{R}^N}\frac{|f(z+y) - f(z)|}{\alpha}\right), \quad x,y \in \mathbb{R}^N,$$

and

Corollary 1 (Regularity) *Let u be a solution of*

$$\mathbb{F}(\mathrm{D}^2u,u,x)+f=0 \quad \text{in } \mathbb{R}^{\mathrm{N}}$$

where f uniformly continuous in $\mathbb{R}^{\mathrm{N}}$ and $\mathbb{F}(\mathrm{X},r,x)\equiv\mathbb{F}(\mathrm{X},r)$, $a_0(x)\equiv\alpha$, then

$$\rho_u(s)\le\beta^{-1}\left(\frac{\rho_f(s)}{\alpha}\right),\quad s\ge 0,$$

where by ρ_h we denote the modulus of continuity of a function h. □

As it was pointed out above, the behavior at infinity of the subsolutions of (2) in the whole space $\mathbb{R}^{\mathrm{N}}$ is not known a priori. Under (3), (4), (5) and (9) we may give an estimate by arguing on the cubes $\mathbf{Q}_{\frac{|x|}{2}}(x)$. More precisely, it follows

$$\limsup_{|x|\to\infty}\frac{u^*(x)-\beta^{-1}\left(\sup\limits_{\mathbf{Q}_{\frac{|x|}{2}}(x)}\dfrac{f_+}{\alpha}\right)}{\Psi^{-1}\left(c(\mathrm{A}_\infty)\sqrt{\dfrac{\alpha}{\mathrm{N}\Lambda}}\dfrac{|x|}{2}\right)}\le\frac{\mathrm{N}}{\mathrm{A}_\infty}.$$

Remark 4 The assumption (9) is in some sense sharp in the above contributions (see Proposition 1 below). On the other hand, we emphasize, once more, that the above results and their consequences are obtained under the inequality $0\le\lambda\le\Lambda$, $\Lambda>0$, hence $\mathbb{F}$ can be eventually degenerate. □

Let us return to the inequality (14) whenever one satisfies

$$\int_{0^+}\frac{ds}{\sqrt{\mathbb{B}(s)}}<\infty,\quad \mathbb{B}'=\beta. \tag{21}$$

Then

$$\mathrm{R}_0\doteq\sqrt{\frac{\Lambda\mathrm{N}}{\alpha}\frac{\Psi(0)}{c(\mathrm{A}_\infty)}}<\infty$$

implies

$$\mathrm{W}_{\mathrm{R}_0}(x_0)=\frac{\mathrm{N}}{\mathrm{A}_\infty}\Psi^{-1}\big(\Psi(0)\big)=0,\quad \text{if } \mathbf{Q}_{\mathrm{R}_0}(x_0)\subset\Omega,$$

for which a dead core can appear. So by using the classical inequality $\|x\|_2\le\sqrt{\mathrm{N}}\|x\|_\infty$ we obtain

Theorem 3 *Assume* (3), (4), (5), (9) *and* (21). *Let u be a solution of*

$$\mathbb{F}\big(\mathrm{D}^2u,u,x\big)+f=0 \quad \text{in } \Omega\subseteq\mathbb{R}^{\mathrm{N}}.$$

Then

$$u(x)=0 \quad \text{if} \quad \operatorname{dist}(x,\operatorname{supp} f)\ge\mathrm{N}\sqrt{\frac{\Lambda}{\alpha}\frac{\Psi(0)}{c(\mathrm{A}_\infty)}},$$

provided $\operatorname{supp} f\ne\mathbb{R}^{\mathrm{N}}$. □

Remark 5 We note that the localization is independent on the values of u on the eventual boundary $\partial\Omega$. General properties of the dead core have been collected in the monograph [9]. □

Remark 6 Theorem 3 applies for an increasing function on $\mathbb{R}$ such that $\beta(r)=r^m$ with $m>1$ for large r and $0<m<1$ for small r. □

2 Interior properties and other comments

We begin here with a technicality directly derived from (9) for which we may define a function $\Phi(\zeta)$ given implicity by

$$\int_{A\Phi(\zeta)}^{\infty} \frac{ds}{\sqrt{\mathbb{B}(s)}} = \frac{B}{\sqrt{\Lambda}}\zeta \tag{22}$$

where A and B are two positive constants to be chosen later. By means of the auxiliar decreasing function $\Psi :]0,\infty[\to]0,\Psi(0)[$ given by

$$\Psi(t) = \int_t^{\infty} \frac{ds}{\sqrt{\mathbb{B}(s)}}, \quad 0 < t < \infty, \quad \Psi(0) \leq \infty,$$

one has

$$\Psi\big(A\Phi(\zeta)\big) = \frac{B}{\sqrt{\Lambda}}\zeta \quad \Leftrightarrow \quad \Phi(\zeta) = \frac{1}{A}\Psi^{-1}\left(\frac{B}{\sqrt{\Lambda}}\zeta\right), \tag{23}$$

with $\Phi(0) = \infty$, $\Phi\left(\frac{\sqrt{\Lambda}}{B}\Psi(0)\right) = 0$. We note that Ψ is independent on the constants A and B. Then we obtain

$$\begin{cases} -\Phi_{\zeta}(\zeta) = \dfrac{B}{A\sqrt{\Lambda}}\sqrt{\mathbb{B}\big(A\Phi(\zeta)\big)}, \\ \Phi_{\zeta\zeta}(\zeta) = \dfrac{B^2}{2A\Lambda}\beta\big(A\Phi(\zeta)\big), \end{cases} \quad 0 \leq \zeta < \frac{\sqrt{\Lambda}}{B}\Psi(0). \tag{24}$$

So that, we get to the one dimensional result

Lemma 1 *Let* $R_k > 0$ *given in* (10), *with* $N = 1$, *and* $x_0 \in \mathbb{R}$, *then the function*

$$w(x) = \frac{1}{A_\infty}\Psi^{-1}\left(c(A_\infty)\sqrt{\frac{\alpha}{\Lambda}}\frac{R_k^2 - |x-x_0|^2}{R_k}\right), \quad -R_k < x - x_0 < R_k, \tag{25}$$

satisfies

$$-\Lambda w''(x) + a_0(x)\beta\big(w(x)\big) \geq 0, \quad -R_k < x - x_0 < R_k,$$

provided (5) *and* (9).

PROOF. In order to simplify we assume with no loss of generality $x_0 = 0$, then the function

$$w(x) = \Phi\big(\zeta(x)\big), \quad \zeta(x) = R_k^2 - |x|^2, \quad -R_k < x < R_k,$$

satisfies

$$\begin{cases} w'(x) &= \Phi_{\zeta}\big(\zeta(x)\big)\zeta'(x) = 2x\dfrac{B}{A\sqrt{\Lambda}}\sqrt{\mathbb{B}\big(A\Phi\big(\zeta(x)\big)\big)}, \\ w''(x) &= \Phi_{\zeta\zeta}\big(\zeta(x)\big)\big(\zeta'(x)\big)^2 + \Phi_{\zeta}\big(\zeta(x)\big)\zeta''(x) \\ &= 4x^2\Phi_{\zeta\zeta}\big(\zeta(x)\big) + 2\dfrac{B}{A\sqrt{\Lambda}}\sqrt{\mathbb{B}\big(A\Phi\big(\zeta(x)\big)\big)}. \end{cases}$$

We note that

$$w(x) = \Phi\big(R_k^2 - |x|^2\big) = \frac{1}{A}\Psi^{-1}\left(\frac{B}{\sqrt{\Lambda}}\big(R_k^2 - |x|^2\big)\right), \quad -R_k < x < R_k,$$

is well defined if

$$\frac{B}{\sqrt{\Lambda}}R_k^2 \leq \Psi(0).$$

Then $w''(x) \geq 0$ and

$$w''(x) \leq 2R_k^2\frac{B^2}{A\Lambda}\beta\big(\Phi(\zeta)\big) + 2\frac{B}{A\sqrt{\Lambda}}\sqrt{A\Phi\big(\zeta(x)\big)\beta\big(\Phi\big(\zeta(x)\big)\big)}.$$

On the other hand, inequality

$$\frac{\mathrm{B}}{\sqrt{\Lambda}}\mathrm{R}_k^2 \geq \frac{\mathrm{B}}{\sqrt{\Lambda}}\zeta(x) \geq \int_{\mathrm{A}\Phi(\zeta(x))}^{\Phi(\zeta(x))} \frac{ds}{\sqrt{\mathbb{B}(s)}} \geq \frac{(1-\mathrm{A})\Phi\big(\zeta(x)\big)}{\sqrt{\mathbb{B}\big(\Phi\big(\zeta(x)\big)\big)}} \geq \frac{(1-\mathrm{A})\Phi\big(\zeta(x)\big)}{\sqrt{\Phi\big(\zeta(x)\big)\beta\big(\Phi\big(\zeta(x)\big)\big)}}$$

implies

$$\sqrt{\mathrm{A}\Phi\big(\zeta(x)\big)\beta\big(\Phi\big(\zeta(x)\big)\big)} \leq \frac{\mathrm{BR}_k^2}{(1-\mathrm{A})}\sqrt{\frac{\mathrm{A}}{\Lambda}}\beta\big(\Phi\big(\zeta(x)\big)\big),$$

whence

$$\Lambda w''(x) \leq 2\mathrm{B}^2\mathrm{R}_k^2\left[\frac{1}{\mathrm{A}} + \frac{1}{\sqrt{\mathrm{A}}(1-\mathrm{A})}\right]\beta\big(w(x)\big) \leq a_0(x)\beta\big(w(x)\big), \quad -\mathrm{R}_k < x < \mathrm{R}_k,$$

provided

$$\left(\frac{\mathrm{BR}_k}{c(\mathrm{A})}\right)^2 = \alpha.$$

In fact, for $\mathrm{A} = \mathrm{A}_\infty$ and these choice of B one has

$$\frac{\mathrm{B}}{\sqrt{\Lambda}}\mathrm{R}_k^2 = \Psi(k) \leq \Psi(0).$$

□

A reasoning of separation of variables with respect to addition enables us to extend the results of Lemma 1 to arbitrary dimension $\mathrm{N} \geq 1$.

PROOF OF THEOREM 1 Again we assume with no loss of generality $x_0 = 0$. Then the reasoning of the proof of Lemma 1 proves that the function

$$\mathrm{W}_{\mathrm{R}_k}(x) = \sum_{i=1}^{\mathrm{N}} \Phi\big(\zeta(x_i)\big), \quad \zeta(x_i) = \mathrm{R}_k^2 - |x_i|^2, \quad -\mathrm{R}_k < x_i < \mathrm{R}_k,$$

satisfies, in the cube $\mathbf{Q}_{\mathrm{R}_k}(0)$, the property

$$\begin{aligned}\Lambda\Delta\mathrm{W}_{\mathrm{R}_k}(x) \quad &\leq 2\mathrm{B}^2\mathrm{R}_k^2\left[\frac{1}{\mathrm{A}_\infty} + \frac{1}{(1-\mathrm{A}_\infty)\sqrt{\mathrm{A}_\infty}}\right]\sum_{i=1}^{\mathrm{N}}\beta\big(\Phi\big(\zeta(x_i)\big)\big)\\ &\leq 2\mathrm{NB}^2\mathrm{R}_k^2\left[\frac{1}{\mathrm{A}_\infty} + \frac{1}{(1-\mathrm{A}_\infty)\sqrt{\mathrm{A}_\infty}}\right]\beta\big(\mathrm{W}_{\mathrm{R}_k}(x)\big)\end{aligned}$$

or

$$\Lambda\Delta\mathrm{W}_{\mathrm{R}_k}(x) \leq a_0(x)\beta\big(\mathrm{W}_{\mathrm{R}_k}(x)\big)$$

provided

$$\mathrm{BR}_k = c(\mathrm{A}_\infty)\sqrt{\frac{\alpha}{\mathrm{N}}}.$$

So that, since $\mathrm{D}^2\mathrm{W}_{\mathrm{R}_k}(x) \geq 0$, in the $\mathscr{S}^{\mathrm{N}}$ sense, we have $\mathscr{P}^+_{\lambda,\Lambda}\big(\mathrm{D}^2\mathrm{W}_{\mathrm{R}_k}(x)\big) = \Lambda\Delta\mathrm{W}_{\mathrm{R}_k}(x)$ (see (6)) and consequently

$$-\mathscr{P}^+_{\lambda,\Lambda}\big(\mathrm{D}^2\mathrm{W}_{\mathrm{R}_k}(x)\big) + a_0(x)\beta\big(\mathrm{W}_{\mathrm{R}_k}(x)\big) \geq 0, \quad x \in \mathbf{Q}_{\mathrm{R}_k}(0),$$

and the inequalities

$$\begin{cases}\mathbb{F}(\mathrm{D}^2u^*, u^*, x) + f \geq 0 \quad \text{in } \mathbf{Q}_{\mathrm{R}_k}(0),\\ \mathbb{F}(\mathrm{D}^2(v_* + \mathrm{W}_{\mathrm{R}_k}), v_* + \mathrm{W}_{\mathrm{R}_k}, x) + g \leq 0 \quad \text{in } \mathbf{Q}_{\mathrm{R}_k}(0).\end{cases}$$

hold. Since $\mathrm{W}_{\mathrm{R}_k} = \infty$ on $\partial\mathbf{Q}_{\mathrm{R}_k}(0)$ we deduce

$$u^* \leq v_* + \mathrm{W}_{\mathrm{R}_k} \quad \text{on } \partial\mathbf{Q}_{\mathrm{R}_k-\varepsilon}(0),$$

for small $\varepsilon > 0$, then comparison principle (see [4] or [5]) and (4) lead to

$$u^* \leq v_* + \mathrm{W}_{\mathrm{R}_k} + \beta^{-1}\left(\sup_{\mathbf{Q}_{\mathrm{R}_k}(0)} \frac{(f-g)_+}{\alpha}\right) \quad \text{in } \mathbf{Q}_{\mathrm{R}_k-\varepsilon}(0).$$

Finally, sending $\varepsilon \to 0$ one concludes the result. □

The importance of the condition (9) in our contribution is shown as follows

Proposition 1 *Under* (3), (4) *and* (5) *let us assume that* (9) *does not hold. Then for each positive constant* M *there exists an unbounded function* $u \in \mathscr{C}^2(\mathbb{R}^{\mathrm{N}})$ *such that*

$$\begin{cases} \mathbb{F}\big(\mathrm{D}^2 u, u, x\big) \leq 0 & in\ \mathbb{R}^{\mathrm{N}}, \\ \mathrm{M} \leq u(x), & x \in \mathbb{R}^{\mathrm{N}}. \end{cases}$$

PROOF. Fixed M we consider the function w given by

$$\int_{\mathrm{M}}^{w(r)} \frac{ds}{\sqrt{\int_{\mathrm{M}}^{s} \beta(\tau)d\tau}} = \sqrt{\frac{\alpha}{2\Lambda}}r, \quad r > 0.$$

Clearly, one has

$$\begin{cases} \Lambda w''(r) = \alpha\beta\big(w(r)\big), & r > 0, \\ w(r) \geq w(0) = \mathrm{M}, \\ w'(0) = 0. \end{cases}$$

Moreover

$$\int^{\infty} \frac{ds}{\sqrt{\mathbb{B}(s)}} = \infty, \quad \mathbb{B}' = \beta,$$

implies

$$\lim_{r\to\infty} w(r) = \infty.$$

Then we define the function

$$u(x) = w\big(|x_1|\big), \quad x = \big(x_1, \ldots, x_{\mathrm{N}}\big) \in \mathbb{R}^{\mathrm{N}},$$

for which the structure assumptions give

$$\mathbb{F}(\mathrm{D}^2 u, u, x) \leq \mathscr{P}^+_{\lambda,\Lambda}(\mathrm{D}^2 u) + \mathbb{F}(0, u, x) \leq \Lambda w''\big(|x_1|\big) - a_0(x)\beta\big(w(|x_1|\big) \leq 0.$$

□

In particular, M being arbitrary, the proof of Proposition 1 shows that if (9) fails is not possible interior estimates for the semilinear equation

$$-\Lambda\Delta u + \alpha\beta(u) = 0.$$

3 Solutions in the whole space

Some consequences of Theorem 1 were included in the Section 2. Now we show that the universal bound (13) and the comparison (19) enable us to give an existence result by means of the Perron method (we send to [4] or [5] for details). More precisely

Theorem 4 (Existence) *Assume* (3), (4), (5) *and* (9). *Let* $f \in \mathscr{C}\big(\mathbb{R}^{\mathrm{N}}\big)$ *bounded. Then the equation*

$$\mathbb{F}\big(\mathrm{D}^2 u, u, x\big) + f = 0 \quad in\ \mathbb{R}^{\mathrm{N}}$$

has a unique bounded continuous solution.

PROOF. We only sketch the proof because we apply the reasoning of [8]. Since $f \in \mathscr{C}\left(\mathbb{R}^N\right)$ is bounded from below there exists a positive constant C such that

$$-\alpha\beta(C) \leq f(x), \quad x \in \mathbb{R}^N,$$

then

$$\mathbb{F}(0,C,x) + f(x) \geq 0, \quad x \in \mathbb{R}^N,$$

implies that

$$\mathscr{G} = \left\{w \text{ upper semi–continuous in } \mathbb{R}^N : \ \mathbb{F}\left(\mathrm{D}^2 w, w, x\right) + f \geq 0 \text{ in } \mathbb{R}^N\right\}$$

is a non empty set. The version of the local universal bounds (13) for semi–continuous solutions of

$$\mathbb{F}\left(\mathrm{D}^2 w, w, x\right) + f \geq 0 \quad \text{ in } \mathbb{R}^N$$

enables us to construct the locally bounded function

$$u(x) = \sup\ \{w(x) :\ w \in \mathscr{G}\}, \quad x \in \mathbb{R}^N. \tag{26}$$

As in [8, Proposition 1], one proves that u is, in fact, the maximal element of the set $\mathscr{G}$, therefore

$$u(x) \leq u^*(x) \leq u(x), \quad x \in \mathbb{R}^N,$$

i.e, $u \equiv u^*$ and

$$\mathbb{F}\left(\mathrm{D}^2 u, u, x\right) + f \geq 0 \quad \text{ in } \mathbb{R}^N.$$

On the other hand, by a contradiction argument, as in [8, Theorem 2], one derives that u_* solves

$$\mathbb{F}\left(\mathrm{D}^2 u_*, u_*, x\right) + f \leq 0 \quad \text{ in } \mathbb{R}^N.$$

Finally, the version of the comparison (19) for semi–continuous solutions concludes

$$u(x) \leq u_*(x), \quad x \in \mathbb{R}^N,$$

whence the result follows. □

Remark 7 Obviously, the solution obtained in (26) has the global transferred regularity given by Corollary 1. □

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