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Velocity fluctuations in laminar fluid flow

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ABSTRACT

Fluctuating hydrodynamics, originally developed for fluctuations in fluids in equilibrium, can be extended to deal also with thermally excited hydrodynamic fluctuations in non-equilibrium states. After first reviewing some results earlier obtained for temperature fluctuations in fluids subjected to an externally imposed temperature gradient, we use in this paper fluctuating hydrodynamics to determine the enhancement of velocity fluctuations in laminar fluid flow. Adopting the case of planar Couette flow as a representative example, we show how the fluctuations of the wall-normal component of the velocity and of the wall-normal component of the vorticity can be obtained as solutions of a stochastic Orr–Sommerfeld equation and a stochastic Squire equation, respectively. By solving these fluctuating hydrodynamic equations we obtain quantitative estimates of the flow-induced non-equilibrium enhancements of the velocity and vorticity fluctuations as a function of the Reynolds number and of the wave number of the fluctuations.

1. Introduction

The theory of fluctuations in fluids that are in thermodynamic equilibrium is well developed [1], with a theoretical framework for thermally excited fluctuations in fluids in thermodynamic equilibrium states being provided by Landau's fluctuating hydrodynamics [2,3]. More recently, it has been demonstrated that fluctuating hydrodynamics can be extended to deal with thermally excited fluctuations in fluids in stationary non-equilibrium states, predicting non-equilibrium enhancements of the fluctuations [4]. Moreover, while fluctuations in equilibrium are generally spatially short ranged on any hydrodynamic scale except for states near a critical point, fluctuations in fluids in non-equilibrium states turn out to be always long ranged, even far away from any hydrodynamic instability. As a consequence, non-equilibrium fluctuations are affected by the presence of boundaries.

In previous work we have examined in considerable detail temperature and concentration fluctuations in fluids and fluid mixtures subjected to an externally imposed temperature gradient as reviewed in [4]. The purpose of the present paper is to elucidate how fluctuating hydrodynamics can be used to determine thermally excited velocity fluctuations in laminar fluid flow. To illustrate the method of fluctuating hydrodynamics we shall consider specifically laminar flow in a fluid layer bounded by two horizontal plates, commonly referred to as planar Couette flow [5,6]. Many investigators have studied the effect of external perturbations on laminar flow [7–11]. Fluctuating hydrodynamics enables us to determine the intrinsic velocity fluctuations that are always present, even in the absence of any externally imposed noise. As shown recently [12,13], the intrinsic wall-normal velocity and vorticity fluctuations can be deduced by solving stochastic modifications of the well-known deterministic Orr–Sommerfeld and Squire equations, respectively. In this paper we shall further analyze the physical properties of these solutions without repeating the somewhat intricate mathematic details of the solution procedures. We shall make a quantitative comparison between the two contributions to the fluctuations and further investigate the anisotropic nature of the intrinsic velocity and vorticity fluctuations in laminar flow.

We shall proceed as follows. In Section 2 we briefly review the method of fluctuating hydrodynamics and its extension to non-equilibrium states using a one-component fluid in a temperature gradient as an example. In Section 3 we initiate the application of fluctuating hydrodynamics to treat velocity fluctuations in laminar flow and specify the stochastic Orr–Sommerfeld and Squire equations for the wall-normal velocity and vorticity fluctuations, respectively. In Section 4 we discuss the solution of the stochastic Orr–Sommerfeld equation for the wall-normal velocity fluctuations, and in Section 5 the solution of the stochastic Squire equations for the wall-normal vorticity fluctuations. We conclude with a discussion of the results in Section 6.

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2. Fluctuating hydrodynamics

2.1. Fluctuating temperature equation

In fluctuating hydrodynamics one introduces fluctuating dissipative fluxes in the balance equations for momentum and energy [2–4]. For instance, from the balance of energy one obtains for the change of the temperature T as a function of the time t at constant pressure p a balance equation of the form

$$\rho c_p \left[\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right] = -\nabla\cdot\mathbf{Q}. \tag{1}$$

Here ρ is the mass density, c_p the isobaric specific heat capacity, $\mathbf{v}$ the fluid velocity and $\mathbf{Q}$ the heat flux, which is written as

$$\mathbf{Q} = -\lambda\nabla T + \delta\mathbf{Q}, \tag{2}$$

where λ is the thermal-conductivity coefficient. That is, the linear phenomenological laws for the thermodynamic fluxes, like Fourier's law for heat conduction here, are only satisfied "on average" and to be supplemented by a fluctuating flux $\delta\mathbf{Q}$ such that on average $\langle\delta\mathbf{Q}\rangle = 0$.

Substitution of Eq. (2) into Eq. (1) yields a "fluctuating" equation for the temperature:

$$\rho c_p \left[\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T\right] = \lambda\nabla^2 T - \nabla\cdot\delta\mathbf{Q}. \tag{3}$$

To determine the thermally excited fluctuations one writes

$$T = T_0 + \delta T(\mathbf{r}, t), \tag{4}$$

and

$$\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}, \tag{5}$$

where T_0 is the average local temperature and $\mathbf{v}_0$ the average local flow velocity.

2.2. Temperature fluctuations in thermodynamic equilibrium

In thermodynamic equilibrium T_0 is a constant independent of location $\mathbf{r}$ and time t, while $\mathbf{v}_0 = 0$. Substituting the expressions (4) and (5) for the fluctuating temperature and velocity in Eq. (3) and only retaining terms linear in the fluctuations one obtains

$$\rho c_p \frac{\partial \delta T}{\partial t} = \lambda\nabla^2\delta T - \nabla\cdot\delta\mathbf{Q}. \tag{6}$$

The equilibrium correlation functions for the components δQ_i of the fluctuating heat flux $\delta\mathbf{Q}$ are given by the fluctuation–dissipation theorem [2–4,14]:

$$\langle\delta Q_i(\mathbf{r}, t)\cdot\delta Q_j(\mathbf{r}', t')\rangle = 2k_B\lambda T^2\delta_{ij}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{7}$$

where k_B is Boltzmann's constant and T=T_0. Upon taking the spatial Fourier transforms of Eqs. (6) and (7), one readily obtains for the time-dependent correlation function of the temperature fluctuations as a function of the wave number q:

$$\langle\delta T^*(\mathbf{q}, t)\cdot\delta T(\mathbf{q}', t')\rangle = S_T^{\mathrm{E}}\exp(-a_T q^2|t-t'|)(2\pi)^3\delta(\mathbf{q}-\mathbf{q}'), \tag{8}$$

where $a_T \equiv \lambda/\rho c_p$ is the thermal diffusivity and where S_T^{E} is the intensity of the equilibrium temperature fluctuations which is independent of the wave number q:

$$S_T^{\mathrm{E}} = \frac{k_B T^2}{\rho c_p}. \tag{9}$$

2.3. Temperature fluctuations in non-equilibrium

To elucidate the extension of fluctuating hydrodynamics to non-equilibrium stationary states we consider as an example a fluid layer with height L confined between two horizontal plates at different stationary temperatures. This arrangement is commonly referred to as the Rayleigh–Bénard problem whose dynamic features are governed by the Rayleigh number Ra defined as [15]

$$\mathrm{Ra} = \frac{\alpha_p L^4\mathbf{g}\cdot\nabla T_0}{\nu a_T}, \tag{10}$$

where α_p is the thermal expansion coefficient, ν the kinematic viscosity, $\mathbf{g}$ the gravitational force, and ∇T_0 the imposed temperature gradient.

We first consider the case in which the plates are heated from above, so that Ra is negative. For any negative value of Ra the fluid remains hydrodynamically stable without any macroscopic fluid motion. Then the average local temperature $T_0(\mathbf{r})$ in Eq. (4) depends linearly on the z-coordinate in the wall-normal direction, but the average fluid velocity $\mathbf{v}_0$ in Eq. (5) continues to be zero. It then follows from Eq. (3) that Eq. (6) for the temperature fluctuations changes into:

$$\rho c_p \left[\frac{\partial \delta T}{\partial t} + \delta\mathbf{v}\cdot\nabla T_0\right] = \lambda\nabla^2\delta T - \nabla\cdot\delta\mathbf{Q}. \tag{11}$$

We see that the second term on the left-hand side of Eq. (11) causes now a coupling between the temperature fluctuations and the velocity fluctuations through the presence of the temperature gradient ∇T_0. Hence, in addition to Eq. (11), we also need to consider the fluctuating Navier–Stokes equation for $\delta\mathbf{v}$ at constant pressure which reads in first approximation

$$\frac{\partial\delta\mathbf{v}}{\partial t} = \nu\nabla^2\delta\mathbf{v} + \frac{1}{\rho}\nabla\cdot\delta\Pi, \tag{12}$$

where $\delta\Pi$ is a fluctuating stress tensor whose equilibrium correlation functions for its components $\delta\Pi_{ij}$ are given by [2–4,14]

$$\langle\delta\Pi_{ij}(\mathbf{r}, t)\cdot\delta\Pi_{kl}(\mathbf{r}', t')\rangle = 2k_B T_0\rho\nu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{13}$$

while any cross-correlations between the components of $\delta\mathbf{Q}$ and $\delta\Pi$ are absent. In the fluctuating Navier–Stokes equation (12) we have for the moment neglected a contribution from gravity and assumed $\nabla\cdot\delta\mathbf{v} = 0$. There has been some confusion of the meaning of this assumption in the context of fluctuating hydrodynamic [16]. The incompressible-flow assumption is commonly adopted in fluid dynamics, since it is expected to be valid when the fluid velocity is small compared to the speed of sound [2]. However, when this assumption is combined with the expression (5) for the fluctuating velocity, it follows that in the limit of zero velocity $\nabla\cdot\delta\mathbf{v} = 0$, which implies that only two of the three components of the fluid velocity can fluctuate independently [13]. Thus in the context of fluctuating hydrodynamics use of the incompressible-flow assumption strictly assumes an incompressible fluid [17].

Since the fluid is still at rest, one might naively expect that the temperature fluctuations still could be given by Eq. (8), but now with the local-equilibrium values of S_T^{E} and a_T. However, because of the coupling between temperature and velocity fluctuations which is absent in equilibrium, this strong local-equilibrium assumption is not valid. Instead one adopts a weaker local-equilibrium assumption, namely that the correlation functions for the fluctuating dissipative fluxes (also referred to as noise terms) are still given by their equilibrium expressions (7) and (13) with the local-equilibrium values for λ, ρ, ν, and $T = T_0$ [18]. Although these local-equilibrium properties depend on the temperature, it has been found that in practice they can be approximated by their average values in the fluid layer [19,20]. From Eqs. (11) and (12) it follows that the temperature fluctuations will contain two dynamic modes, a heat mode with a decay rate determined by the thermal diffusivity a_T and a viscous mode with a decay rate determined by the kinematic viscosity ν [18,21,22]. However, in this paper we shall restrict our attention to the intensity of the fluctuations given by the equal-time correlation functions.

For the intensity of the temperature fluctuations in a fluid subjected to a temperature gradient ∇T_0 one obtains an expression of the form:

$$\langle \delta T^*(\mathbf{q},t)\delta T(\mathbf{q}',t)\rangle \equiv S_T(\mathbf{q},\mathbf{q}') = S_T^{\mathrm{E}}[1+\Delta S_T^{\mathrm{NE}}(\mathbf{q})](2\pi)^3\,\delta(\mathbf{q}-\mathbf{q}'), \tag{14}$$

where $\Delta S_T^{\mathrm{NE}}(\mathbf{q})$ represents a non-equilibrium enhancement of the temperature fluctuations, which depends on the wave vector $\mathbf{q}$ as

$$\Delta S_T^{\mathrm{NE}}(q) = S_0^{\mathrm{NE}}\frac{\tilde{q}_\parallel^2}{\tilde{q}^6}. \tag{15}$$

In this equation $\tilde{q} = qL$ is a dimensionless wave number, $\tilde{q}_\parallel$ is the magnitude of the component of $\mathbf{q}$ parallel to the horizontal plates, while the coefficient S_0^{NE} accounts for the dependence of the non-equilibrium enhancement on the (local) values of the physical properties and the magnitude of the temperature gradient:

$$S_0^{\mathrm{NE}} = \frac{(\mathrm{Pr}-1)(c_p/T)L^4}{\nu^2 - a_T^2}(\nabla T_0)^2, \tag{16}$$

where $\mathrm{Pr} = \nu/a_T$ is the Prandtl number. We note that, apart from a negligible contribution from an adiabatic temperature gradient, S_0^{NE} is equal to the symbol $\tilde{S}_0^{\mathrm{NE}}$ in [23], but differs from the symbol $\tilde{S}_0^{\mathrm{NE}}$ in [24] by a factor $\mathrm{Pr}+1$. From Eq. (15) we see that the non-equilibrium enhancement of the temperature fluctuations attains its largest value for wave vectors $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_\parallel$ in the direction perpendicular to the temperature gradient so that for such fluctuations

$$\Delta S_T^{\mathrm{NE}}(q) = S_0^{\mathrm{NE}}\frac{1}{\tilde{q}^4}, \qquad (\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_\parallel,\, q\to\infty). \tag{17}$$

From Eqs. (16) and (17) we conclude that the non-equilibrium enhancement of the temperature fluctuations will be proportional to the square of the temperature gradient and inversely proportional to the fourth power of the wave number. This dependence of the non-equilibrium enhancement on both temperature gradient and wave number has been verified experimentally with high accuracy [19,25,26].

Eqs. (15) and (17) represent the enhancement in the limit of large wave numbers ($q \to \infty$). For decreasing value of the wave number one needs to retain a gravitational term in the fluctuating Navier–Stokes Eq. (12)[27]. Moreover, because of the long-ranged nature of the non-equilibrium temperature fluctuations encompassing the entire vertical spatial dimension of the fluid layer [23,28], the fluctuations are affected by boundary conditions. The presence of boundaries breaks down the translational invariance of the system along the vertical z-direction. Consequently, because of the presence of boundaries, two-point correlation functions of thermodynamic variables are no longer proportional to delta functions $\delta(q_z - q_z')$, like in Eq. (14). Difficulties caused by this fact can be overcome by averaging over the height of the layer, ($q_z \simeq 0$), to obtain correlation functions with wave vectors $\mathbf{q}_\parallel$ in the horizontal plane, in which plane they are not only translationally invariant but, in the case of the Rayleigh–Bénard problem, also isotropic.

An analysis of these two-point correlation functions in the horizontal plane shows that the main effect of the boundary conditions is to cause the non-equilibrium enhancement to vanish proportionally to q^2 in the limit $q \to 0$, with an amplitude that depends on the actual boundary conditions. In the case of somewhat less realistic stress-free boundary conditions, the fluctuating equations (11) and (12) can be solved analytically and one obtains for $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_\parallel$ [23]:

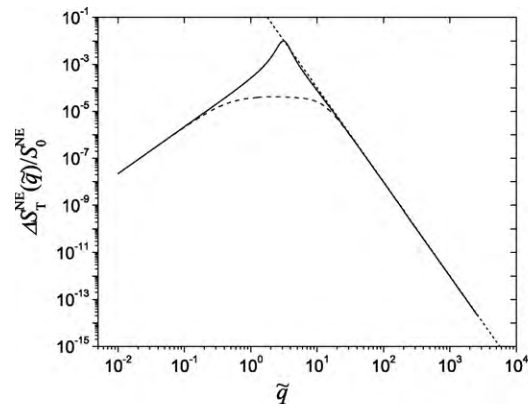

Fig. 1. Normalized non-equilibrium enhancement $\Delta S_T^{\mathrm{NE}}(\mathbf{q})/S_0^{\mathrm{NE}}$ of the temperature fluctuations with wave vector $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_\parallel$ as a function of $\tilde{q}$ at Ra = +1650 (solid curve) and Ra = −25, 000 (dotted curve) for Pr = 5. The plots represent a first-order Galerkin approximation for rigid-boundary conditions [24]. The dotted line represents the asymptotic $1/q^4$ solution for large wave numbers.

$$\Delta S_T^{\mathrm{NE}}(q) = S_0^{\mathrm{NE}}\frac{17}{20160}\tilde{q}^2, \quad (\tilde{\mathbf{q}}=\tilde{\mathbf{q}}_\parallel,\, q\to 0 \text{ for stress-free boundaries}). \tag{18}$$

For more realistic rigid boundaries we have obtained an estimate in a first-order Galerkin approximation [24]:

$$\Delta S_T^{\mathrm{NE}}(q) \simeq S_0^{\mathrm{NE}}\frac{3(\mathrm{Pr}+1)}{896(21\mathrm{Pr}+5)}\tilde{q}^2, \qquad (\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_\parallel,\, q\to 0 \text{ for rigid boundaries}), \tag{19}$$

which depends somewhat on the value of the Prandtl number Pr. Hence, the non-equilibrium enhancement as a function of the wave number q exhibits a crossover from a q^{-4} dependence for large q to a q^2 dependence for small q. While the limiting behavior for large and small q is independent of the Rayleigh number Ra, the non-equilibrium enhancement at intermediate values of q strongly depends on Ra. To illustrate the wave-number dependence we show in Fig. 1 plots of the normalized non-equilibrium enhancement $\Delta S_T^{\mathrm{NE}}(q)/S_0^{\mathrm{NE}}$ at $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_\parallel$ as a function of $\tilde{q}$ for a negative and a positive value of Ra. For negatives values of Ra (i.e., when the fluid layer is heated from above) the non-equilibrium enhancement exhibits a broad maximum at intermediate values of $\tilde{q}$, a feature that has been confirmed experimentally for the similar case of non-equilibrium concentration fluctuations in a concentration gradient [29]. For positive values of Ra (i.e., when the fluid layer is heated from below) the maximum enhancement of the nonequilibrium temperature fluctuations increases rapidly as Ra approaches a critical value $\mathrm{Ra_c}$ and one recovers the known limiting behavior [30,31] of the temperature fluctuations asymptotically close to the onset of convection at $\mathrm{Ra} = \mathrm{Ra_c}$ and $\tilde{q} = \tilde{q}_\mathrm{c}$ [4].

3. Application of fluctuating hydrodynamics to velocity fluctuations

Having validated fluctuating hydrodynamics for describing fluctuations of physical properties in fluids in stationary non-equilibrium states as discussed in the preceding section, we are now ready to apply the same method to evaluate the non-equilibrium enhancement of velocity fluctuations in laminar fluid flow. For this purpose we consider the simplest case, namely that of a liquid under incompressible laminar flow (thus with uniform density ρ) between two horizontal boundaries separated by a distance $2L$ as indicated schematically in Fig. 2. This arrangement is commonly

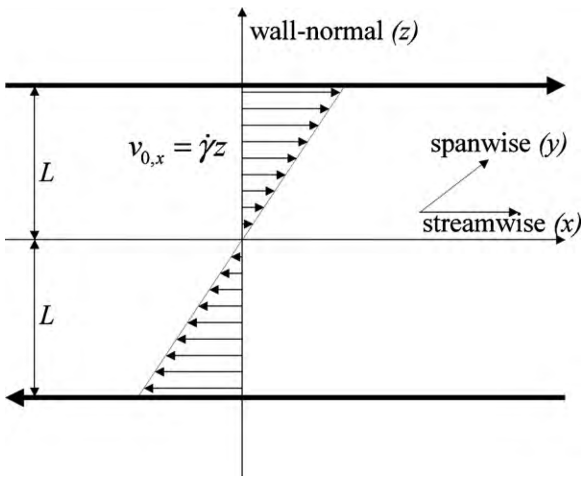

Fig. 2. Schematic representation of planar laminar Couette flow.

referred to as planar Couette flow or plane Couette flow. We adopt a coordinate system with the X-axis in the streamwise direction, the Y-axis in the spanwise direction, and the Z-axis in the wall-normal direction [5,11]. The average fluid velocity $\mathbf{v}_0 = \{v_x, v_y, v_z\} = \{\dot{\gamma}z, 0, 0\}$ in the X direction in Eq. (5) depends on the vertical coordinate z with a constant shear rate $\dot{\gamma}$. Since the average velocity $\mathbf{v}_0$ is no longer zero, the Navier–Stokes equation (12) for the fluctuating velocity $\delta\mathbf{v}$ changes into [4,32]:

$$\frac{\partial \delta\mathbf{v}}{\partial t} + \dot{\gamma} z \frac{\partial \delta\mathbf{v}}{\partial x} + \dot{\gamma}\hat{x}\delta v_z = -\frac{1}{\rho}\nabla\delta p + \nu\nabla^2\delta\mathbf{v} + \frac{1}{\rho}\nabla\cdot\delta\Pi, \tag{20}$$

where $\hat{x}$ is the unit vector in the streamwise direction. In obtaining Eq. (20) we have continued to adopt the incompressibility assumption $\nabla\cdot\delta\mathbf{v} = 0$ as was done in the preceding section.

It is convenient to introduce dimensionless variables defined by

$$\tilde{r} = \frac{r}{L}, \quad \tilde{q} = qL, \quad \tilde{t} = \dot{\gamma}t, \quad \delta\tilde{p} = \frac{\delta p}{\rho L^2 \dot{\gamma}^2},$$
$$\delta\tilde{\mathbf{v}} = \frac{\delta\mathbf{v}}{L\dot{\gamma}}, \quad \delta\tilde{\Pi} = \frac{\delta\Pi}{\rho L^2\dot{\gamma}^2}, \tag{21}$$

so that Eq. (20) can be rewritten as

$$\frac{\partial \delta\tilde{\mathbf{v}}}{\partial \tilde{t}} + \tilde{z}\frac{\partial \delta\tilde{\mathbf{v}}}{\partial \tilde{x}} + \hat{x}\delta\tilde{\mathbf{v}}_z = -\tilde{\nabla}\delta\tilde{p} + \frac{1}{\mathrm{Re}}\tilde{\nabla}^2\delta\tilde{\mathbf{v}} + \tilde{\nabla}\cdot\delta\tilde{\Pi}, \tag{22}$$

where Re is the Reynolds number of the flow:

$$\mathrm{Re} = \frac{L^2\dot{\gamma}}{\nu}. \tag{23}$$

For the correlation function for the components of the fluctuating stress tensor we adopt the local equilibrium version of Eq. (13), which in terms of the dimensionless components $\tilde{\Pi}_{ij}$ become:

$$\langle \delta\tilde{\Pi}_{ij}(\tilde{\mathbf{r}}, \tilde{t})\cdot\delta\tilde{\Pi}_{kl}(\tilde{\mathbf{r}}', \tilde{t}')\rangle = 2\frac{k_B T_0 \nu}{\rho\dot{\gamma}^3 L^7}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}')\delta(\tilde{t} - \tilde{t}'). \tag{24}$$

In principle the local temperature T_0 exhibits a shallow parabolic profile as a function of the vertical coordinate z as a result of viscous heating [33]. However, previous work [20] has indicated that the effect of any height dependence of temperature on the intensity of the fluctuations is very small and we take the temperature T_0 to be uniform and independent of the height, as is usually done in the literature dealing with planar Couette flow.

In the remainder of this paper we shall only use dimensionless variables and to simplify the notation we shall from now on delete the tildes indicating dimensionless variables explicitly. From Eq. (22) we see that there is no coupling between velocity fluctuations and temperature fluctuations, but there is a potential coupling between velocity fluctuations and pressure fluctuations. However, for a incompressible fluid with $\nabla\cdot\delta\mathbf{v} = 0$ the pressure term can be eliminated by taking a double rotational of Eq. (22) so that one obtains for the fluctuations δv_z of the wall-normal velocity component v_z [12]

$$\frac{\partial\nabla^2\delta v_z}{\partial t} + z\frac{\partial\nabla^2\delta v_z}{\partial x} - \frac{1}{\mathrm{Re}}\nabla^4\delta v_z = \{\nabla\times\nabla\times\nabla\cdot\delta\Pi\}_z. \tag{25}$$

As mentioned earlier, in an incompressible fluid only two velocity components can fluctuate independently. As the second fluctuating component it is convenient to consider the fluctuations $\delta\omega_z$ of the wall-normal component of the vorticity:

$$\delta\omega_z = \frac{\partial\delta v_x}{\partial y} - \frac{\partial\delta v_y}{\partial x}. \tag{26}$$

An equation for the wall-normal-vorticity fluctuations is obtained by taking a single curl of Eq. (22), so that

$$\frac{\partial\delta\omega_z}{\partial t} + z\frac{\partial\delta\omega_z}{\partial x} - \frac{\partial\delta v_z}{\partial y} - \frac{1}{\mathrm{Re}}\nabla^2\delta\omega_z = \{\nabla\times\nabla\cdot\delta\Pi\}_z. \tag{27}$$

On the left-hand sides of Eqs. (25) and (27) one recognizes the well-known Orr–Sommerfeld equation and the well-known Squire equation in the fluid-dynamics literature for the wall-normal velocity v_z and the wall-normal vorticity ω_z [5,6]. Hence, fluctuating hydrodynamics shows that the intrinsic velocity and vorticity fluctuations that are always present, even in the absent of any external perturbations, can be obtained by solving a stochastic Orr–Sommerfeld Eq. (25) for δv_z and then solving a stochastic Squire Eq. (27) for $\delta\omega_z$. The correlation functions for the noise terms on the right-hand sides of Eqs. (25) and (27) can readily be deduced from Eq. (24).

4. Wall-normal velocity fluctuations in laminar flow

A procedure for solving the stochastic Orr–Sommerfeld Eq. (25), without incorporating boundary conditions, has been presented by Lutsko and Dufty [34,35]. The solution for the wall-normal velocity fluctuations can be written in the form

$$\langle \delta v_z^*(\mathbf{q}, t)\cdot\delta v_z(\mathbf{q}', t)\rangle \equiv C_z(\mathbf{q}, \mathbf{q}') = C_z^{\mathrm{E}}[1 + \Delta C_z^{\mathrm{NE}}(\mathbf{q})](2\pi)^3\delta(\mathbf{q} - \mathbf{q}'), \tag{28}$$

where C_z^{E} represents the intensity of the fluctuations of an individual velocity component in an incompressible fluid in the absence of flow [12] and where $\Delta C_z^{\mathrm{NE}}(\mathbf{q})$ is the enhancement of the intensity of the wall-normal velocity fluctuations in the presence of flow. The non-equilibrium enhancement of the temperature fluctuations, discussed in Section 2, originated from two coupled fluctuating hydrodynamic equations, that is, from a coupling of two hydrodynamic modes, namely one mode associated with temperature fluctuations and another mode associated with the fluctuations of the velocity at the same value of the wave number q. The non-equilibrium enhancement of the wall-normal velocity fluctuations arises from a single fluctuating hydrodynamic equation, but the term $z(\partial\nabla^2\delta v_z/\partial x)$ in Eq. (25) causes a coupling between fluctuations of the same hydrodynamic mode with different values of the wave number q, as has earlier been pointed out by Lutsko and Dufty [34] and by Wada and Sasa [36].

To elucidate the anisotropic nature of the non-equilibrium enhancement of the velocity fluctuations and vorticity fluctuations, we specify the wave vectors $\mathbf{q} = \{q_x, q_y, q_z\}$ and $\mathbf{q}_\parallel = \{q_x, q_y, 0\}$ in terms of spherical coordinates with the polar angle θ measured from the Z axis in the wall-normal direction and the azimuthal

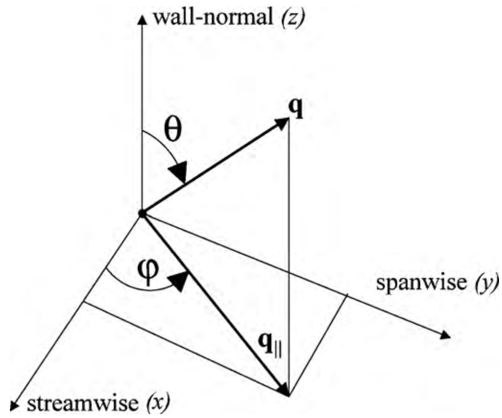

Fig. 3. Spherical coordinates for representing the wave vectors $\mathbf{q}$ and $\mathbf{q}_\parallel$.

angle φ in the XY plane such that $\varphi = 0$ corresponds to the streamwise direction and $\varphi = \pi/2$ to the spanwise direction as indicated in Fig. 3. Thus

$$q_z = q\cos\theta, \qquad q_\parallel = q\sin\theta, \qquad q_x = q\sin\theta\cos\varphi, \qquad q_y = q\sin\theta\sin\varphi. \tag{29}$$

For large wave numbers q we obtain [12]

$$\Delta C_z^{\rm NE}(q) = \frac{\rm Re}{2q^2}\cos\varphi\sin 2\theta - \frac{\rm Re^2}{2q^4}\cos^2\varphi\cos 2\theta\sin^2\theta \quad (q\to\infty), \tag{30}$$

which does not depend on any boundary conditions. Eq. (30) shows that the non-equilibrium enhancement of the wall-normal velocity fluctuations is highly anisotropic. We first note that the enhancement vanishes when $q_\parallel = 0$ ($\theta = 0$), that is, when the wave vector is in the direction of the shear gradient, just as the enhancement (15) of the temperature fluctuations vanishes when the wave vector is in the direction of the temperature gradient. The first term in (30) is proportional to q^{-2}, which corresponds to a r^{-1} dependence in real space. This term is the one found by Tremblay et al. [32] and Lutsko and Dufty [34] but it actually vanishes when $\mathbf{q} = \mathbf{q}_\parallel$ ($\theta = \pi/2$). Instead for $\mathbf{q} = \mathbf{q}_\parallel$ the enhancement of the wall-normal velocity fluctuations becomes

$$\Delta C_z^{\rm NE}(\mathbf{q}) = \frac{\rm Re^2}{2q^4}\cos^2\varphi \qquad (\mathbf{q} = \mathbf{q}_\parallel, q\to\infty), \tag{31}$$

which agrees with the asymptotic behavior found by Wada and Sasa [36].

Just as in Eq. (17), the dependence on q^{-4} implies long-ranged correlations over macroscopic length scales, so that for smaller q the enhancement is affected by the boundary conditions. As was the case for a fluid under a temperature gradient, inclusion of boundary conditions breaks down translational invariance along the wall-normal direction [12]. Again, we can average over the height of the layer to obtain two-point correlation functions with wave vector $\mathbf{q}_\parallel$ on the horizontal plane, where they are translationally invariant. However, in contrast with the case of the Rayleigh–Bénard problem, for planar Couette flow two-point velocity correlations continue to be anisotropic in the parallel plane. In a previous publication, we have evaluated the enhancement of the wall-normal-velocity fluctuations with no-slip boundary conditions in a second-order Galerkin approximation, and found in the limit of small wave numbers [12]

$$\Delta C_z^{\rm NE}(\mathbf{q}) \simeq \frac{2\rm Re^2}{2079}q^2\cos^2\varphi \qquad (\mathbf{q} = \mathbf{q}_\parallel, q\to 0). \tag{32}$$

Hence, just like the non-equilibrium enhancement of the temperature fluctuations in a temperature gradient, also the enhancement of the wall-normal-velocity fluctuations in laminar flow exhibits a crossover from a q^{-4} dependence for large q to a q^2 dependence for small q. From Eqs. (31) and (32) we see that the enhancement of the wall-normal velocity fluctuations vanishes when the wave vector $\mathbf{q}_\parallel = \{0, q_y\}$ is in the spanwise direction ($\varphi = \pi/2$) and has its maximum when the wave vector $\mathbf{q}_\parallel = \{q_x, 0\}$ is in the streamwise direction ($\varphi = 0$). We also note from Eqs. (31) and (32) that the enhancement in both limits is proportional to the square of the Reynolds number. In Fig. 4 we have plotted the normalized enhancement $\Delta C_z^{\rm NE}(\mathbf{q})/\mathrm{Re}^2$ for $\mathbf{q}_\parallel = \{q_x, 0\}$ in the streamwise direction as a function of q_x for two values of the Reynolds number, Re = 20 and Re = 300, obtained in a second-order Galerkin approximation [12]. The structure of the enhancement at intermediate values of the wave number at the larger value of Re is thought to be related to a confluent singularity proportional to $q^{-4/3}$, first noticed by Lutsko and Dufty [35], and recovered by Wada and Sasa [36] and by us [12].

5. Wall-normal vorticity fluctuations in laminar flow

A procedure for solving the stochastic Squire equation (27) has been presented in [13]. In analogy to Eq. (28) the solution can be written in the form

$$\langle\delta\omega_z^*(\mathbf{q},t)\cdot\delta\omega_z(\mathbf{q}',t)\rangle \equiv W_z(\mathbf{q},\mathbf{q}') = W_z^{\rm E}[1+\Delta W_z^{\rm NE}(\mathbf{q})](2\pi)^3\delta(\mathbf{q}-\mathbf{q}'), \tag{33}$$

where $W_z^{\rm E}$ represents the intensity of an individual vorticity component of an incompressible fluid in the absence of flow and where $\Delta W_z^{\rm NE}(\mathbf{q})$ is the enhancement of the wall-normal vorticity fluctuations in the presence of flow. One can notice a difference in the structure of the stochastic Squire Eq. (27) when compared with the stochastic Orr–Sommerfeld Eq. (25). First, the term $z(\partial\delta\omega_z/\partial x)$ in Eq. (27), just like the term $z(\partial\nabla^2\delta v_z/\partial x)$ in Eq. (25), causes a coupling of the hydrodynamic mode associated with the vorticity fluctuations with different wave numbers q; we refer to this coupling mechanism as "self coupling". In addition the term $\partial\delta v_z/\partial y$ in Eq. (27) causes a cross coupling between the vorticity fluctuations and the velocity fluctuations. It turns out that, unlike the case of the stochastic Orr–Sommerfeld equation (25), the self coupling in the stochastic Squire equation (27) reproduces the intensity of the vorticity fluctuations in the absence of flow and does not contribute to any enhancement of these fluctuations in the presence of flow. Hence, the enhancement $\Delta W_z^{\rm NE}(\mathbf{q})$ in Eq. (34) arises solely form the cross coupling between vorticity and velocity fluctuations. For

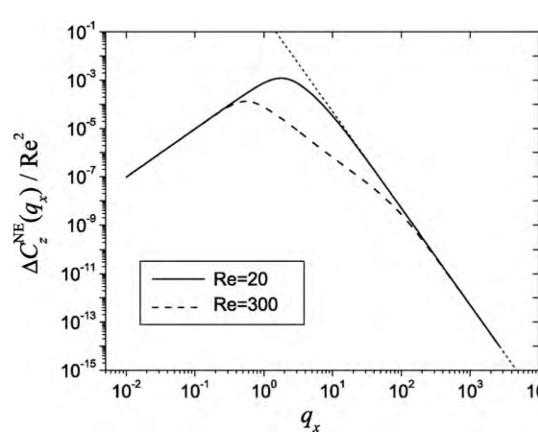

Fig. 4. Normalized enhancement $\Delta C_z^{\rm NE}(\mathbf{q})/\mathrm{Re}^2$ of the wall-normal-velocity fluctuations with dimensionless wave vectors $\mathbf{q} = \mathbf{q}_\parallel = \{q_x, 0, 0\}$ in the streamwise direction as a function q_x, at Re=20 and Re=300. The dotted line represents the asymptotic $1/2q^4$ solution for large wave numbers.

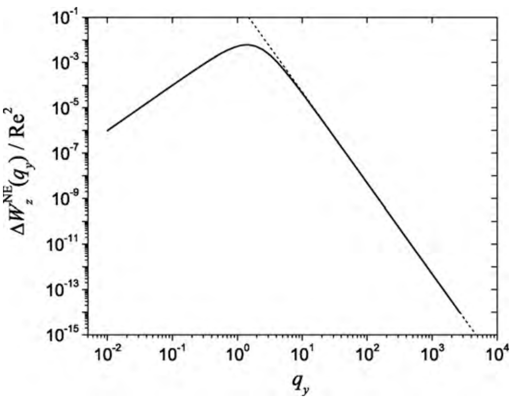

Fig. 5. Normalized enhancement $\Delta W_z^{\rm NE}(\mathbf{q})/\mathrm{Re}^2$ of the wall-normal-vorticity fluctuations with dimensionless wave vectors $\mathbf{q}=\mathbf{q}_\parallel=\{0, q_y, 0\}$ in the spanwise direction as a function q_y. The dotted line represents the asymptotic $1/2q^4$ solution for large wave numbers.

$\mathbf{q}=\mathbf{q}_\parallel$ we obtain in the limit of large q [13]

$$\Delta W_z^{\rm NE}(\mathbf{q}) = \frac{\mathrm{Re}^2}{2q^4}\sin^2\varphi \qquad (\mathbf{q}=\mathbf{q}_\parallel, q\to\infty), \tag{34}$$

independent of the boundary conditions. Again, the incorporation of boundary conditions lead us to study two-point vorticity correlations with wave vector $\mathbf{q}_\parallel$ on the horizontal plane, after averaging over the height of the layer [13]. In that case, it was found that for small parallel q, in a second-order Galerkin approximation [13]

$$\Delta W_z^{\rm NE}(\mathbf{q}) \simeq \frac{3\mathrm{Re}^2}{364}q^2\sin^2\varphi \qquad (\mathbf{q}=\mathbf{q}_\parallel, q\to 0). \tag{35}$$

From Eqs. (34) and (35) we see that the enhancement of the wall-normal vorticity fluctuations vanishes when the wave vector $\mathbf{q}_\parallel=\{q_x, 0\}$ is in the streamwise direction ($\varphi=0$) and has its maximum when the wave vector $\mathbf{q}_\parallel=\{0, q_y\}$ is in the spanwise direction ($\varphi=\pi/2$), just opposite to what we found in the preceding section for the wall-normal velocity fluctuations. In Fig. 5 we show a plot of the normalized enhancement $\Delta W_z^{\rm NE}(\mathbf{q})/\mathrm{Re}^2$ for $\mathbf{q}_\parallel=\{0, q_y\}$ in the spanwise direction as a function of q_y. In this case the enhancement is proportional to Re^2 for all values of the Reynolds number, so that the normalized enhancement $\Delta W_z^{\rm NE}(\mathbf{q})/\mathrm{Re}^2$ is independent of Re.

Both the enhancement of the velocity fluctuations and of the vorticity fluctuations are anisotropic functions of the wave vector $\mathbf{q}_\parallel$. In Fig. 4 we showed the enhancement of the wall-normal-velocity fluctuations for wave vectors $\mathbf{q}=\mathbf{q}_\parallel$ in the streamwise direction and in Fig. 5 we showed the enhancement of the wall-normal-vorticity fluctuations for wave vectors $\mathbf{q}=\mathbf{q}_\parallel$ in the spanwise direction. In Fig. 6 we show contours of constant actual enhancements $\Delta C_z^{\rm NE}(\mathbf{q})$ and $\Delta W_z^{\rm NE}(\mathbf{q})$ for wave vectors $\mathbf{q}=\mathbf{q}_\parallel=\{q_x, q_y\}$ in the first quadrant of the XY plane for Re = 100. We recall that in first approximation the enhancements are proportional to the square of the Reynolds number as seen from Figs. 4 and 5. We note from the labels of the constant enhancement curves in Fig. 6 that the enhancement $\Delta W_z^{\rm NE}(\mathbf{q})$ caused by a cross coupling of velocity and vorticity fluctuations is generally more significant than $\Delta C_z^{\rm NE}(\mathbf{q})$ arising from self coupling. It is also possible to evaluate the non-equilibrium energy amplification resulting from these fluctuations and again one finds that the dominant contribution to the non-equilibrium energy amplification is the one resulting from the cross coupling between velocity and vorticity fluctuations [13].

We remark that the main effect of thermal noise in plane Couette flow is to enhance vorticity fluctuations in the plane parallel to the walls with a spanwise modulation given by $q_y \simeq 1.6$, corresponding to the maximum observed in Fig. 5 and in the right panel of Fig. 6. These vorticity fluctuations, of course, have to be added to the average velocity $\mathbf{v}_0$, which is in the streamwise direction. The overall effect will produce very elongated rolls or streaks, consistent with a spanwise modulation of the streamwise velocity. It is interesting to note that the appearance of streaks is the first step in a generic nonlinear unstabilization process proposed by Waleffe [37] for laminar flows. This self-sustained process initiated by streaks has been used to predict the generic existence of traveling waves in laminar flows, a prediction that was subsequently observed experimentally [38].

6. Discussion

In this paper we have provided strong evidence that the method of fluctuating hydrodynamics [2,3], originally developed for fluctuations in fluids in equilibrium, can be extended to deal with fluctuations in fluids in stationary non-equilibrium states. After reviewing previous results obtained for temperature fluctuations in a fluid subjected to a stationary temperature gradient, we have

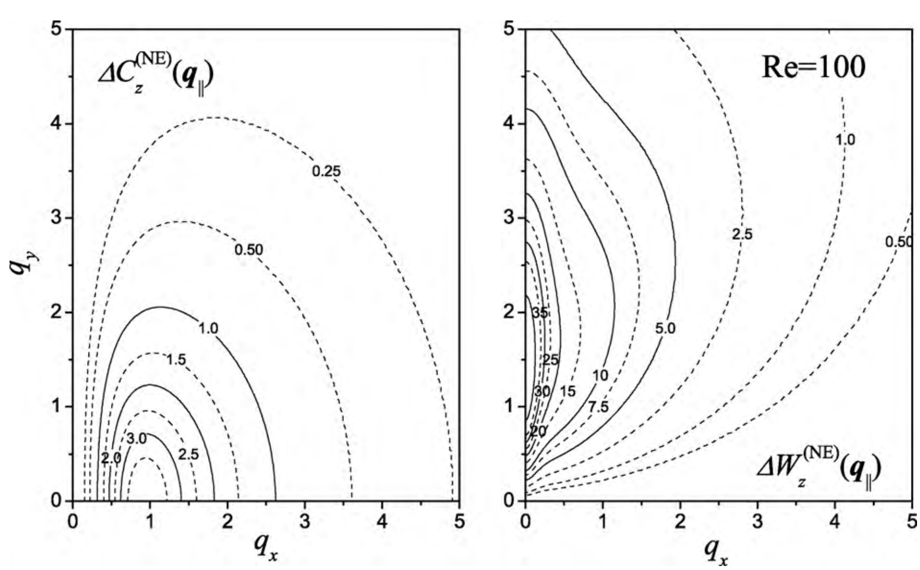

Fig. 6. Contours of constant enhancements $\Delta C_z^{\rm NE}(\mathbf{q})$ and $\Delta W_z^{\rm NE}(\mathbf{q})$ of the wall-normal-velocity and wall-normal-vorticity fluctuations, respectively, for dimensionless wave vectors $\mathbf{q}=\mathbf{q}_\parallel=\{q_x, q_y\}$ in the XY plane, at Re = 100.

shown how fluctuating hydrodynamics can also be used to derive the enhancement of velocity and vorticity fluctuations in laminar fluid flow illustrating the method for the case of isothermal planar Couette flow. We have found the presence of two mode-coupling mechanisms in laminar flow, namely self coupling of the same hydrodynamic modes and cross coupling between velocity and vorticity fluctuations. We have found that the cross coupling is the most significant mechanism contributing to the enhancement of the fluctuations in laminar flow.

The enhancement of the fluctuations is anisotropic and depends on the direction of the wave vector $\mathbf{q}$. Furthermore, in contrast to equilibrium states, the fluctuations are spatially long ranged, even far away from any hydrodynamic instability. For wave vectors $\mathbf{q} = \mathbf{q}_{\parallel}$ perpendicular to either the temperature gradient in a non-isothermal fluid or to the shear gradient in laminar flow we find that the enhancements of the fluctuations exhibit a crossover from a q^{-4} dependence for large q to a q^2 dependence for small q, as shown in Figs. 1, 4, and 5.

The non-equilibrium enhancement of the temperature fluctuations in fluids subjected to a temperature gradient depends on the wave number q and on the Rayleigh number Ra as shown in Fig. 1. The maximum enhancement increases with increasing values of Ra and diverges at a critical value $\mathrm{Ra_c}$ corresponding to the onset of thermal convetion. However, the enhancement of the velocity and vorticity fluctuations increases with increasing values of the Reynolds number Re, roughly as Re^2, but remains finite for any finite value of Re. Hence, like traditional hydrodynamic-instability analysis investigating the effects of externally imposed perturbations [5,6,9], also the intrinsic fluctuations predicted by fluctuating hydrodynamics do not yield a critical value of the Reynolds number that can be associated with a transition from laminar to turbulent flow. To solve the problem of the transition from laminar to turbulent flow other physical mechanisms will have to be considered [9,39,38].

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