Mini-course: Computer-Assisted Proofs in Nonlinear Analysis

Instructor: **Olivier Hénot** (École Polytechnique, Paris)

Schedule: May 21st, 22nd, 23rd, 10h00-12h00

Venue: Seminario 209 (aula Alberto Dou), F. de Matematicas, UCM

Organizers: J. M. Arrieta (jarrieta@ucm.es) and P. Lappicy (philemos@ucm.es)

This introductory mini-course is aimed at advanced undergraduates, master's and PhD students. Researchers with an interest in the topic are also welcome.

Overview: Nonlinear differential equations are essential to modern science, offering a framework for modeling time-dependent phenomena and capturing a wide range of complex behaviors. As these models have grown in complexity, numerical simulations have become indispensable tools for their analysis, particularly for systems in infinite-dimensional function spaces, such as those governed by partial differential equations. However, despite their importance, numerical methods come with inherent limitations. Approximations – whether from rounding errors or discretization – introduce uncertainties, and often without offering a posteriori error bounds. These limitations raise critical questions about the reliability of numerical results, particularly in chaotic systems, where even minor computational inaccuracies can lead to drastically different outcomes. The need for more rigorous, mathematically grounded computational techniques has never been more pressing in the pursuit of understanding these complex systems.

In response to these challenges, computer-assisted proofs have emerged as a compelling approach, blending traditional analysis and computational methods. In recent years, such techniques have seen increasing success, with notable examples including the proof of Jones' conjecture [1], the proof of finite time blow-up for the compressible Euler and Navier-Stokes equations [2] and the proof of Marchal's conjecture [3]. For further context, see the survey articles [4, 5]. The mini-course is organized as follows:

Module 1 (May 21st, 10h-12h): Finite-dimensional problems
Concepts: contraction mapping theorem and Newton-Kantorovich, introduction to interval arithmetic

Module 2 (May 22nd, 10h-12h): Algebraic equations
Concepts: an infinite-dimensional problem, deriving a rigorous inverse operation using Taylor series

Module 3 (May 23rd, 10h-12h): Periodic orbits
Concepts: periodic solutions to ordinary differential equations using Fourier series

Parts of the lectures will include hands-on exercises using the Julia programming language. Participants are strongly encouraged to **bring their own laptops** and to install both Julia and the `RadiiPolynomial` library in advance. For step by step instructions, see:

`https://olivierhnt.github.io/Computer-assisted-proofs-in-nonlinear-analysis/installation/`

References

[1] J. Jaquette. A proof of Jones' conjecture. *Journal of Differential Equations*, 266(6):3818–3859, 2019.

[2] G. Cao-Labora, J. Gómez-Serrano, J. Shi, and G. Staffilani. Non-radial implosion for compressible Euler and Navier-Stokes in $\mathbb{T}^3$ and $\mathbb{R}^3$. *ArXiv*, 2023.

[3] R. Calleja, C. García-Azpeitia, O. Hénot, J.-P. Lessard, and J. D. Mireles James. From the lagrange triangle to the figure eight choreography: proof of Marchal's conjecture. *ArXiv*, 2024.

[4] J. Gómez-Serrano. Computer-assisted proofs in PDE: a survey. *SeMA Journal*, 76:459–484, 2019.

[5] J. B. van den Berg and J.-P. Lessard. Rigorous numerics in dynamics. *Notices of the American Mathematical Society*, 62(9), 2015.