Cooperation, Imitation and Partial Rematching

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Abstract

We study a setting where imitative players are matched into pairs to play a Prisoners' Dilemma game. A well known result in such setting is that under random matching cooperation vanishes for any interior initial condition. The novelty of this paper is that we consider partial rematching: players that belong to a pair where both parties cooperate repeat partner next period whilst all other players are randomly matched into pairs. This rematching mechanism makes cooperation the unique outcome in the long run under some conditions. Furthermore, we show that if imitation happens infrequently enough then cooperative behavior is always present in the population.

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1 Introduction

Individuals learn by imitation when their choices are based on the alternatives they observe others choose. The inability of real life subjects to correctly understand and process all the information they dispose of is a common justification for the use of imitation in some economic models. For example, in a situation where many players interact with each other, correctly anticipating other agents' actions can be a massive computational burden.

A well known property of imitation is that, under some conditions, it rules out dominated actions[1]. Thus, if every period imitative players are randomly matched to play a Prisoner's Dilemma game, cooperation vanishes. Given the importance of cooperation and its constant presence in societies and the relevance of imitation for modelling bounded rational behavior (see, for example, Axelrod (1984), Banerjee (1992), Eshel et al (1998) or Ellison and Fudenberg (1995)), the question we raise is: can cooperation survive when players learn by imitation?

We answer this question by exploring the mechanism by which players are matched to play a Prisoners' Dilemma game. The novelty of this paper is that certain rematching is introduced: players who cooperated with each other last period meet again in the next period whilst the rest of players are randomly matched into pairs. This matching mechanism is inspired by a simple rule of thumb: no player should have incentives to keep a non-cooperative partner. Examples of situations where this matching mechanisms seems plausible range from dealing with business suppliers to academic co-authorship or dating.

In the results of this paper, three main conclusions are achieved: First, under some conditions and for any interior initial condition, the survival of cooperation is guaranteed. That is, the situation where no player cooperates is not stable if some conditions on the payoff matrix and/or the specific imitative rule employed are satisfied. Second, no assortative rest point exists. This means that, apart from the rest points on the boundaries, a situation where cooperative players do not face non-cooperative ones is not a rest point. Finally, we find that cooperation is more likely to prevail if imitation happens infrequently. In the limit this means that for all payoff matrices there exists a probability of imitating below which some level of cooperation is always present in the long run.

The reason behind the survival of cooperation lies in the fact that the matching mechanism considered in this paper adds a positive externality to playing cooperatively: in a situation where two players cooperate, switching action has the disadvantage that next period a new opponent, who might not be so keen on playing cooperatively, is faced. Thus, players that cooperate may enjoy more payoff over time than these not cooperating. In this situation,

[1]See, for instance, Schlag (1998) Remark 6.

non-cooperative players imitate cooperative ones, making the survival of cooperation possible.

To our knowledge, only Levine and Pesendorfer (2007), Bergstrom (2003) and Bergstrom and Stark (1993) study similar settings to the one considered in this paper. Levine and Pesendorfer (2007) show that cooperation can survive within a population who learns by imitation if each player holds some information about the strategy of the player with whom she is matched. Bergstrom (2003) and Bergstrom and Stark (1993) proves conditions under which cooperation survives in an evolutionary model where players are either cooperators or defectors, and are more likely to face a player of their same type.

The difference between this paper and Levine and Pesendorfer (2007) lies in that in our model there is a set of matches that are anonymous whilst in Levine and Pesendorfer (2007) all matches are non-anonymous. The present paper differs from Bergstrom and Stark (1993) and Bergstrom (2003) in that players can change their actions from one period to another. Thus, in our model, playing cooperatively in the present period is no guarantee of exhibiting a cooperative behavior in the next period.

The issue of partner selection in cooperative games has recently attracted attention from experimental economists. Duffy and Ochs (2009) conduct an experiment where a Prisoners Dilemma game with two treatments is considered. In the first treatment, matching is completely random whereas in the second one each player always repeats partner. The authors find that cooperation does not emerge in the random matching setting while it does in the fixed pairs treatment. Yang et al (2007) present an experiment where a Prisoner Dilemma game is played and individuals with similar histories are more likely to be matched together. Their results show that cooperation has a higher chance of survival when a history-dependent correlation is added to the matching process. Grimm and Mengel (2009) develop an experiment where players choose between two Prisoner's Dilemma games that differ in the gains from defection. Choosing the game with lower gains signals the player's willingness to cooperate. Grimm and Mengel find that this self selection significantly increases the amount of cooperation.

In order to get a better understanding on cooperation a preferential partner selection, we carry robustness checks and extensions to our main model. In particular, alternative matching processes are considered as extensions to the main model; apart from the matching mechanism whereby only cooperative pairs are maintained, we discuss the cases of complete assortative matching (cooperators only meet cooperators, defectors only meet defectors), all pairs are kept with some fixed probability and, finally, a setting where correlation is not perfect (only a fraction of cooperative pairs are maintained from one period to another). We argue in which of these settings cooperation is more likely to be sustained in the long run and in which ones cooperation does not survive.

The rest of the paper is organized as follows. In Section 2, we develop the model. Section 3 presents the main analysis and the results. In Section 4, we present a further comparison with the literature, a discussion on our assumptions, and some extensions. Finally, Section 5 concludes.

2 The Model

Assume an infinite an even population of players. At the beginning of each period $t = 0, 1, 2, \ldots$, every player is paired with another one and plays the following symmetric stage game against her partner:

Table 1: The Stage Game

	C	D
C	R, R	S, T
D	T, S	P, P

where C stands for cooperate and D stands for defect. The stage game above has the standard Prisoners' Dilemma structure: $T > R > P > S$ with $T, R, P, S \in \mathbb{R}$.

After the stage game is played, all pairs where at least one player chose D are broken while the rest of pairs are maintained. After that, unpaired players are randomly matched into pairs. The distribution of pairs at the beginning of period $t = 0$ is given.

Let σ_{AB} be the fraction of players who played $A \in \{C, D\}$ last period and faced an opponent who played $B \in \{C, D\}$. In an abuse of notation, we say that a player belongs to σ_{AB} if she played $A \in \{C, D\}$ last period and faced an opponent who played $B \in \{C, D\}$. Given this, we have that $\sigma_{CC} \in [0, 1]$, $\sigma_{CD} = \sigma_{DC} \in [0, 1/2]$ and $\sigma_{DD} = 1 - \sigma_{CC} - 2\sigma_{CD}$. Notice that the fraction of players who maintain partner equals σ_{CC}.

Players follow very simple decision rules. In particular, they observe the action and payoff of a random individual[2] and base their choice of action for the stage game on this information plus the information from own action and payoff. All players in the population are equally likely to be observed.

Let $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \in [0, 1]$ be the probability with which player i changes action if she, who played action $a_i \in \{C, D\}$ and obtained payoff $\pi_i \in \mathbb{R}$, observes player j, who

[2]Since we are dealing with an infinite population, results presented in this paper do not depend on how many players are observed.

chose action $a_j \in \{C, D\}$ and achieved payoff $\pi_j \in \mathbb{R}$. Some assumptions on P are needed for the analysis:

Assumptions.

1. *$P(\{i, a_i, \pi_i\}\{j, a_i, \pi_j\}) = 0$,*

2. *$P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) > 0$ if and only if $\pi_i < \pi_j$ and $a_i \neq a_j$ and,*

3. *for all $i, j \in [0, 1]$ and all $a_i, a_j \in A$:*

   - *if $\pi_j > \pi'_j$ then $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \geq P(\{i, a_i, \pi_i\}\{j, a_j, \pi'_j\})$,*
   - *if $\pi_i < \pi'_i$ then $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\}) \geq P(\{i, a_i, \pi'_i\}\{j, a_j, \pi_j\})$.*

The first two assumptions are standard in imitation models (see, for instance, Schlag (1998)). Assumption 1 implies that players change their action only if the player they observe played a different action than the one they chose. Assumption 2 means that there is a positive probability of changing action if and only if observed action yielded more payoff than own action. The third assumption is a monotonicity condition that relates to reinforcement learning models (see, for example, Börgers et al (2004) and Rustichini (1999)). It means that the probability of changing action is weakly increasing in observed payoff and weakly decreasing in own payoff.

We simplify notation when using the function $P(\{i, a_i, \pi_i\}\{j, a_j, \pi_j\})$ as follows: Denote by $P_{AB} : A^2 \times \mathbb{R}^2 \rightarrow [0, 1]$ the probability with which a player in σ_{AB} with $A, B \in \{C, D\}$ changes action. Assumptions $1 - 3$ impose some restrictions on the functional forms of P_{CC}, P_{CD}, P_{DC} and P_{DD}. The function P_{CC} is only positive if the player in σ_{CC} observes a player in σ_{DC} (assumption 1 and 2). In this case, the payoff of observed player equals T while own payoff equals R. Thus, since $\sigma_{DC} = \sigma_{CD}$ we can write P_{CC} as

$$P_{CC} = \sigma_{CD} f(T, R) \tag{1}$$

for some function $f : \mathbb{R}^2 \rightarrow [0, 1]$ given. The two arguments in f are observed payoff and own payoff respectively. The function f is weakly increasing in its first argument and weakly decreasing in its second argument by assumption 3. Furthermore, by assumption 2, $f(\pi', \pi) = 0$ for any $\pi' > \pi$. No other assumptions are needed on f.

The function P_{CD} is only positive if the player in σ_{CD} observes either a player in σ_{DC} or a player in σ_{DD}. If the player observed belongs to σ_{DC}, then observed payoff equals T and own payoff equals S. On the other hand, if the player observed belongs to σ_{DD}, then observed payoff equals P and own payoff equals S. Therefore, using the fact that $\sigma_{DC} = \sigma_{CD}$

we can write

$$P_{CD} \;=\; \sigma_{CD} f(T,S) + \sigma_{DD} f(P,S). \tag{2}$$

The function P_{CD} is never positive as players in σ_{DC} obtained the maximum possible payoff and, hence, by assumption 2 do not change action. Thus,

$$P_{DC} \;=\; 0.$$

Finally, P_{DD} is only positive if the player in σ_{DD} observes a player that belongs to σ_{CC}. In this case, observed payoff equals R while own payoff equals P. Hence, we have that

$$P_{DD} \;=\; \sigma_{CC} f(R,P). \tag{3}$$

Let $\sigma^t_{CC}, \sigma^t_{CD}, \sigma^t_{DC}$ and σ^t_{DD} denote the values of $\sigma_{CC}, \sigma_{CD}, \sigma_{DC}$ and σ_{DD} respectively at each point in time $t = 0, 1, 2, \ldots$ before the stage game is played with $(\sigma^0_{CC}, \sigma^0_{CD}) \in \Omega$ given and $\sigma^0_{DC} = \sigma^0_{CD}$, $\sigma^0_{DD} = 1 - \sigma^0_{CC} - 2\sigma^0_{CD}$. At $t = 0$ and prior to the starting of the game, all players not in σ^0_{CC} are randomly and uniformly matched into pairs. For notational convenience the argument t in the functions P_{CC}, P_{CD}, P_{DC} and P_{DD} is omitted.

A player belongs to σ^{t+1}_{CC} if any of the following two situations occur: Firstly, the player previously belonged to σ^t_{CC} and neither her nor her partner change action. Secondly, either the player belonged to σ^t_{CD} and does not change action or the player belonged to σ^t_{DD} and changes action, and is matched with an opponent playing C that did not belong to σ^t_{CC}. Since the fraction of players choosing C at time $t+1$ that did not belong to σ^t_{CC} is given by $\sigma^t_{CD}(1 - P_{CD}) + \sigma^t_{DD} P_{DD}$ and players in σ^t_{CC} are out of the matching pool as they repeat partner we have that:

$$\sigma^{t+1}_{CC} \;=\; \sigma^t_{CC}(1 - P_{CC})^2 + \frac{\left(\sigma^t_{CD}(1 - P_{CD}) + \sigma^t_{DD} P_{DD}\right)^2}{1 - \sigma^t_{CC}} \tag{4}$$

for $\sigma^t_{CC} \neq 1$. If $\sigma^t_{CC} = 1$ then $\sigma^{t+1}_{CC} = \sigma^t_{CC} = 1$ as all players are cooperating and, thus, no player changes action by assumption 1.

As the fraction of players choosing C in period $t+1$ is given by $\sigma^{t+1}_{CC} + \sigma^{t+1}_{CD} = \sigma^t_{CC}(1 - P_{CC}) + \sigma^t_{CD}(1 - P_{CD}) + \sigma^t_{DD} P_{DD}$, using equation (4) we have that

$$\begin{aligned} \sigma^{t+1}_{CD} \;=\; & \sigma^t_{CC}(1 - P_{CC}) + \sigma^t_{CD}(1 - P_{CD}) + \sigma^t_{DD} P_{DD} \\ & - \sigma^t_{CC}(1 - P_{CC})^2 - \frac{\left(\sigma^t_{CD}(1 - P_{CD}) + \sigma^t_{DD} P_{DD}\right)^2}{1 - \sigma^t_{CC}} \end{aligned} \tag{5}$$

for $\sigma^t_{CC} \neq 1$. If $\sigma^t_{CC} = 1$ then $\sigma^{t+1}_{CD} = \sigma^t_{CD} = 0$.

The values of σ_{DC}^{t+1} and σ_{DD}^{t+1} are given by

$$\begin{aligned}\sigma_{DC}^{t+1} &= \sigma_{CD}^{t+1}, \\ \sigma_{DD}^{t+1} &= 1-\sigma_{CC}^{t+1}-2\sigma_{CD}^{t+1}.\end{aligned}$$

Notice that the state and evolution of the system can be fully characterized by the two variables σ_{CC}^t and σ_{CD}^t and and two equations (4) and (5) via the relations $\sigma_{DC}^t=\sigma_{CD}^t$ and $\sigma_{DD}^t = 1-\sigma_{CC}^t-2\sigma_{CD}^t$. Thus, in what follows we focus the analysis on the two variables σ_{CC}^t and σ_{CD}^t using the variable σ_{DD}^t only when it is notationally convenient.

Define the state space Ω as $\Omega = \{(\sigma_{CC},\sigma_{CD}) \in \mathbb{R}_+^2 : \sigma_{CC}+\sigma_{CD} \in [0,1) \cup \sigma_{CD} \in [0,1/2] \cup (\sigma_{CC},\sigma_{CD}) = (1,0)\}$. Whenever we refer to interior points we mean $(\sigma_{CC},\sigma_{CD}) \in \Omega$ with $\sigma_{CC}+\sigma_{CD} \in (0,1)$. Denote the set of interior points by $\mathring{\Omega}$.

Next we define what a rest point of the model at hands is. Intuitively, a rest point is a situation where the measure of players belonging to each of the sets σ_{CC} and σ_{CD} does not change. Formally:

Definition 1. *A rest point is a point* $(\sigma_{CC},\sigma_{CD}) \in \Omega$ *such that* $\sigma_{CC}^{t+1}=\sigma_{CC}^t$ *and* $\sigma_{CD}^{t+1}=\sigma_{CD}^t$ *whenever* $\sigma_{CC}^t=\sigma_{CC}$ *and* $\sigma_{CD}^t=\sigma_{CD}$.

Definition 2. *An interior rest point is a rest point where* $(\sigma_{CC},\sigma_{CD}) \in \mathring{\Omega}$.

Among all interior rest points it is useful to single out the assortative rest points. An assortative rest point is an interior rest point where a fraction of the population play C against themselves while all other players choose D. That is, in an assortative rest point $\sigma_{CD}=0$ and the population is completely separated between cooperators and defectors.

Definition 3. *An assortative rest point is an interior rest point where* $\sigma_{CD}=0$.

In order to illustrate the behavior of the model we present two simulations, both figures 1 and 2 show the evolution of σ_{CC}, σ_{CD}, σ_{DC} and σ_{DD} for certain parameter values where the function f is given by what is known as the Proportional Imitation Rule (PIR henceforth) with dominant switching rate (Schlag (1998))[3]:

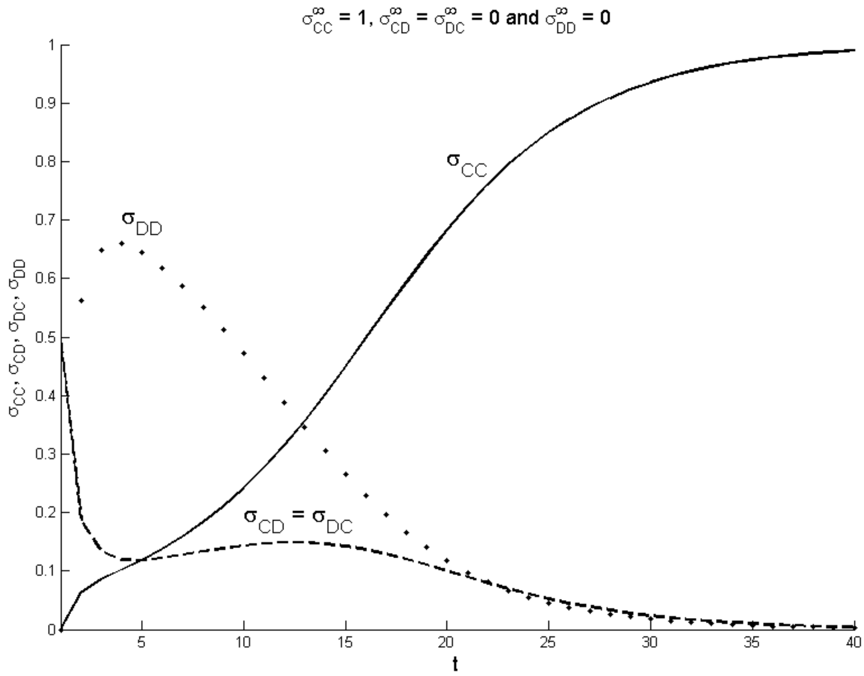
$$f(\pi',\pi) = \frac{1}{T-S}\left(\pi'-\pi\right).$$

As it can be observed in figure 1, during the first periods the amount of cooperative players matched with non-cooperative ones, σ_{CD}, decreases. This is due to the fact that,

[3]A deeper exposition of the relationship between this and other imitation rules and our model is presented in sections 3.3.1, 3.3.2 and 3.3.3.

during these first stages, most cooperative players enjoy less payoff than cooperative ones. However, as times evolves, more and more cooperative players meet each other. After this grouping stage is over, the payoff from cooperating is on average greater than that from not cooperating. This happens because most cooperative players face players that are also cooperative. The level of cooperation increases from there until all players have adapted the cooperative action.

Figure 1: Simulation: PIR with $T = 0.5$, $R = 0.4$, $P = 0$, $S = -0.1$ and $(\sigma^0_{CC}, \sigma^0_{CD}) = (0, 0.5)$



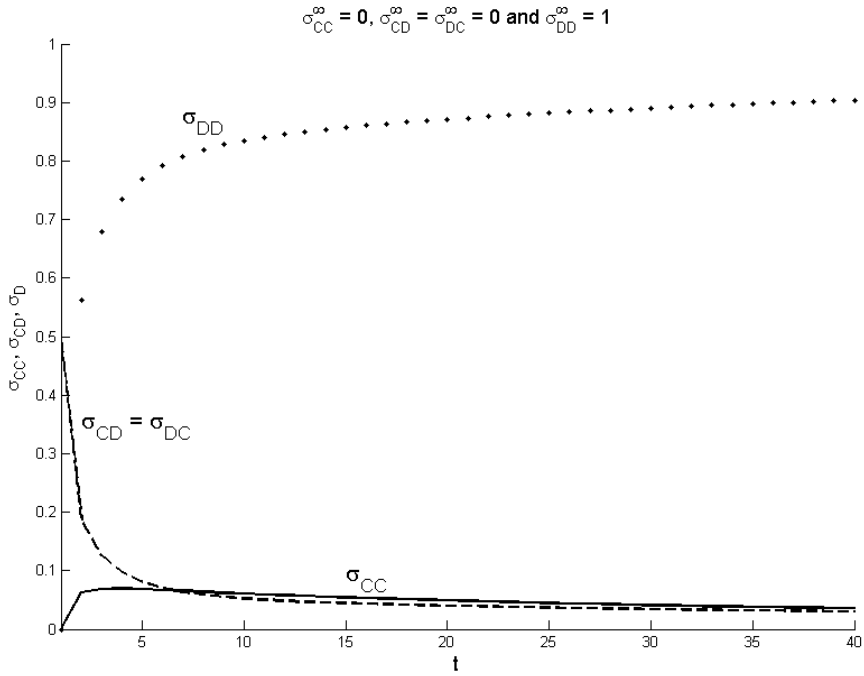
In figure 2, the payoff players in a cooperative pairs get is lower than in the previous simulation. This results in an environment where cooperation vanishes from the population. In figure 2 one can se that the number of players in σ_{CC} initially increases. This is simply due to the fact that some of the players that belong σ_{CD} are matched together and, if they do not change their action immediately, they belong σ_{CC} next period. However, the aggregate level of cooperation decreases until cooperation eventually vanishes from the population.

3 Results

3.1 Random Matching

In this subsection we consider the benchmark case of random matching. Under random matching, all pairs are broken after the stage game is played. We show that, under random

Figure 2: Simulation: PIR with $T = 0.5$, $R = 0.1$, $P = 0$, $S = -0.1$ and $(\sigma^0_{CC}, \sigma^0_{CD}) = (0, 0.5)$



matching, cooperation vanishes for any interior initial condition. The full analysis of the random matching case is presented in the appendix; here we restrict our attention to the main result from this analysis.

Proposition 1. *Under random matching, for any* $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$

$$\lim_{t\to\infty} \sigma^t_{CC} + \sigma^t_{CD} \quad = \quad 0.$$

Proof. See Lemma 1 in the appendix. □

As proposition 1 shows, under random matching cooperation does not survive in the population. This is the known result that under some monotonicity conditions (assumption 3) imitation rules out dominated actions.

With random matching, playing cooperatively is always dominated by the non-cooperative behavior. This is partly because under random matching cooperating has no effect in any period beyond the current one. As we shall see, once we add a certain correlation to the matching process, playing cooperatively may no longer be a dominated action.

3.2 Partial Rematching

We now revert back to the case explained in section 2 where there is partial rematching, i.e. pairs were both players cooperated are maintained in the next period. A first result is that there exist no assortative rest point.

Proposition 2. *No assortative rest point exists.*

Proof. First, note that assuming the necessary and sufficient conditions for a rest point $\sigma_{CC}^{t+1} = \sigma_{CC}^{t}$ and $\sigma_{CD}^{t+1} = \sigma_{CD}^{t}$ and adding equation (4) to equation (5) we obtain

$$\sigma_{DD}P_{DD} - \sigma_{CC}P_{CC} - \sigma_{CD}P_{CD} \quad = \quad 0. \tag{6}$$

The next step is to show that in a rest point with $\sigma_{CD} = 0$ no pairs in σ_{CC} are ever broken. Assume the contrary, this means that some players from σ_{CC} choose D. Hence, if we are at time t then $\sigma_{CC}^{t+1} < \sigma_{CC}^{t}$ unless a set of players in σ_{DD}^{t} switch to C. If this happens, however, we have that some players will be matched against players who chose D in t. Therefore, if a pair is broken, either $\sigma_{CC}^{t+1} < \sigma_{CC}^{t}$ or $\sigma_{CD}^{t+1} > 0$, a contradiction to the definition of assortative rest point.

Given that in an assortative rest point no pairs are ever broken and that $\sigma_{CC} \in (0, 1)$, it follows that all players always choose the same action in the stage game. This implies that players in σ_{CC}^{t} obtain a payoff of R while players in σ_{DD}^{t} obtain a payoff of P. Thus, from assumption 2, it follows that $P_{CC} = 0$ and $P_{DD} > 0$. However, when $P_{CC} = 0$, equation (6) implies that

$$\sigma_{DD}P_{DD} \quad = \quad 0. \tag{7}$$

Since $\sigma_{CC} \in (0, 1)$, $\sigma_{CD} = 0$ and $P_{DD} > 0$, we have that $\sigma_{DD}P_{DD} > 0$, a contradiction to (7). □

The intuition behind the result above is straightforward: In an assortative rest point, cooperative players, σ_{CC}, obtain a payoff of R whilst all the other players, $1 - \sigma_{CC}$, obtain a payoff of $P < R$. Hence, non-cooperative players imitate cooperative ones but cooperative players do not imitate non-cooperative ones. Therefore, the situation with complete separation between cooperators and defectors is not a rest point.

3.3 Local Analysis

The difficulty in obtaining global results lies in the order of the system at hands. As the system in (4) and (5) is of order four, checking for the existence and/or global stability of rest points for a general payoff matrix and an arbitrary function f becomes a highly complex computational task. Thus, we proceed by restricting our attention first to local results.

To start with, we define certain properties of the different rest points. The definitions below are based on Khalil (1995).

Definition 4. *Let $B_r(\sigma_{CC}, \sigma_{CD})$ be the ball of radius $r > 0$ around the point $(\sigma_{CC}, \sigma_{CD}) \in \Omega$. The rest point $(\sigma_{CC}, \sigma_{CD}) \in \Omega$ is*

- *stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(\sigma^0_{CC}, \sigma^0_{CD}) \in \Omega \cap B_\delta(\sigma_{CC}, \sigma_{CD})$ then $(\sigma^t_{CC}, \sigma^t_{CD}) \in \Omega \cap B_\varepsilon(\sigma_{CC}, \sigma_{CD})$ for all $t \geq 0$,*
- *unstable if it is not stable,*
- *asymptotically stable if it is stable and $\delta > 0$ can be chosen such that for any $\kappa < \varepsilon$ if $(\sigma^0_{CC}, \sigma^0_{CD}) \in \Omega \cap B_\delta(\sigma_{CC}, \sigma_{CD})$ then*

$$|| \lim_{t \to \infty} (\sigma^t_{CC}, \sigma^t_{CD}) - (\sigma_{CC}, \sigma_{CD})|| < \kappa,$$

- *a repeller if there exists a $\delta > 0$ such that if $(\sigma^0_{CC}, \sigma^0_{CD}) \in \Omega \cap B_\varepsilon(\sigma_{CC}, \sigma_{CD})$ for all $\varepsilon \in (0, \delta)$ then $(\sigma^t_{CC}, \sigma^t_{CD}) \notin \Omega \cap B_\delta(\sigma_{CC}, \sigma_{CD})$ for some $t \geq 0$,*

We are now ready to state one of the main results of this paper. Namely, if certain conditions on the payoff matrix and/or the specific imitative rule employed are satisfied, then cooperation survives in the long run.

Proposition 3. *If $f(R, P) > 2f(T, R)f(P, S)$, then the rest point $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ is a repeller. On the other hand, if $f(R, P) < 2f(T, R)f(P, S)$, then the rest point $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ is asymptotically stable.*

Proof. See the appendix. □

The idea behind the survival of cooperation under some conditions is the following: Imagine a situation where only a small fraction of players cooperate. Some of these players will be matched together, thus, they repeat partner next period. This set of players playing cooperatively and that are matched together obtain the second-highest payoff, R. Since only very few players cooperate, there is almost no player obtaining the maximum payoff, T. Therefore, under certain conditions, more non-cooperative players imitate cooperative ones than cooperative players imitate non-cooperative ones.

It can be checked that in the simulation performed in figure 1 the condition for $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ to be a repeller is satisfied whilst in the simulation in figure 2 this condition is not satisfied.

Next we seek a better understanding of the conditions in proposition 3 by exploring its implications when different imitative rules are assumed. We focus our attention on three well known such rules: Proportional Imitation Rule, Imitate if Better and Proportional Reviewing Rule.

3.3.1 Proportional Imitation Rule

The general form of the PIR is given by

$$f(\pi', \pi) = s\left(\pi' - \pi\right)$$

where $s \in (0, 1/(T-S)]$ is called the switching rate. The simulation in figures 1 and 2 assumed $s = 1/(T-S)$. This value of the switching rate is known as the dominant switching rate as it leads to the imitation rule that yields a weakly higher expected increase than any other switching rate in the decision maker's payoff in a multi-armed bandit decision problem (Schlag (1998)).

From proposition 3, it is straightforward to show the following characterization for the PIR with dominant switching rate

Corollary 1. *Assume players employ the PIR with switching rate $s = 1/(T-S)$, if $(R-P)(T-S) > 2(T-R)(P-S)$, then the rest point $(\sigma_{CC}, \sigma_{CD}) = (0,0)$ is a repeller.*

Proof. When players employ the PIR with $s = 1/(T-S)$ we can rewrite the condition $(\sigma_{CC}, \sigma_{CD}) = (0,0)$ to be a repeller as

$$(R-P)(T-S) > 2(T-R)(P-S).$$

The result follows. □

To get a better understanding of the result above, we consider a particular case of the payoff matrix of the stage game.

Table 2: The Stage Game - Example

	C	D
C	$\pi_b - \pi_c, \pi_b - \pi_c$	$-\pi_c, \pi_b$
D	$\pi_b, -\pi_c$	$0, 0$

with $1 > \pi_b > \pi_c > 0$. We can interpret π_b as the benefit a player receives when her partner cooperates and π_c as the cost of cooperating. In this case, we have the following result.

Corollary 2. *Assume the stage game is the one given in table 2 and players employ the PIR with switching rate* $s = 1/(\pi_b + \pi_c)$*, if* $\pi_b > \pi_c\sqrt{3}$*, then the rest point* $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ *is a repeller.*

Proof. Comparing the stage games in tables 1 and 2 we have that $T = \pi_b$, $R = \pi_b - \pi_c$, $P = 0$ and $S = -\pi_c$. Using these values in corollary 1 gives the desired result. □

Although $s = 1/(T - S)$ is the dominant switching rate for multi-armed bandit problems, it is the switching rate that is less likely to make cooperation possible in the long run. This can be seen in the condition in proposition 3, the greater the value of f for any given payoff matrix, the less likely the condition for all players to cooperate holds. As a matter of fact, for any payoff matrix, if the switching rate is small enough then cooperation survives for any interior initial condition.

Corollary 3. *Assume players employ the PIR, for any* $T > R > P > S$ *with* $T, R, P, S \in \mathbb{R}$ *there exists a switching rate* $\bar{s}$ *such that for all* $s < \bar{s}$ *we have that the rest point* $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ *is a repeller.*

Proof. When player employ the PIR we can rewrite the condition for the rest point $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ to be a repeller as

$$s(R - P) \quad > \quad 2s^2(T - R)(P - S).$$

Thus, for any $T > R > P > S$ with $T, R, P, S \in \mathbb{R}$ we can choose s small enough so that the inequality above holds true. □

A conclusion that can be drawn from corollary 3 is that when players are more cautious in changing actions then cooperation is more likely to survive in the long run. As we shall see later on, this fact also holds in the limit.

3.3.2 Imitate if Better

Imitate if Better (IB) consists of simply imitating with probability one whenever the action observed yields more payoff than own action. That is, $f(\pi', \pi) = 1$ for all $\pi' > \pi$, $f(\pi', \pi) = 0$ otherwise. We have the following result.

Corollary 4. *Assume players employ IB, then the rest point* $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ *is asymptotically stable.*

Proof. When player employ IB we can rewrite the condition for $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ to be a asymptotically stable as $1 < 2$, which holds trivially. $\square$

As it was already hinted in the previous subsection when the PIR was considered, if the likelihood of imitation is higher, the chances of cooperation to survive are lower. The intuition for the fact that more cautious imitation makes cooperation more likely to survive is that as the benefits from cooperating appear after cooperators repeat partner, if players are more likely to change their actions, then is much less likely that the benefits from repeating partner ever occur. Since these benefits are what makes cooperation possible in our setting, the fact that players change partner more often because of changing action more often makes cooperation harder to sustain. We extend this finding to any imitation rule in proposition 6 in subsection 3.5.

3.3.3 Proportional Reviewing Rule

The Proportional Reviewing Rule (PRR) is similar to the PIR except that own payoff is ignored. That is, the general form of the PRR is given by

$$f(\pi', \pi) = s\pi'$$

with $s \in (0, 1/(T - S)]$. Again, the parameter s is called the switching rate[4].

Corollary 5. *Assume players employ the PRR with switching rate $s = 1/(T - S)$, if $R(T - S) > 2TP$, then the rest point $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ is a repeller.*

Proof. When player employ the PRR we can rewrite the condition for the rest point $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ to be a repeller as

$$R(T - S) > 2TP.$$

$\square$

An immediate consequence from the corollary above is that if players employ the PRR and that the stage game is the one in table 2, then the rest point $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ is a repeller. This is the case since in table 2, $P = 0$ and, thus, under the PRR players that belong to σ_{CD} only imitate those in σ_{DC}. Therefore, since players in σ_{CD} are less likely to change to the non cooperative action, their share in the population increases and eventually they are matched together and, hence, repeat partner next period, i.e. they belong σ_{CC}.

[4]Note that the PRR does not violate assumption 2 as equations (1) - (3) already incorporate this assumption.

3.4 Global Results

As already discussed, the high order of the system we are dealing with makes it complicated to obtain global results. The route we take is to explore the existence of interior rest points. As we shall see later on, if no interior rest point exists then our local results can be extended to global ones.

In order to characterize the existence of interior rest points, we have to examine the rest points of the system given in (4) and (5). We have the following result

Proposition 4. *An interior rest point exists if and only if there exists* $(\sigma_{CC}, \sigma_{DD}) \in \mathring{\Omega}$ *such that*

$$\begin{aligned} \sigma_{CC} P_{CC} + \sigma_{CD} P_{CD} - \sigma_{DD} P_{DD} &= 0, \\ \sigma_{CC} P_{CC} \left(2 - P_{CC}\right) \left(1 - \sigma_{CC}\right) - \left(\sigma_{CD} + \sigma_{CC} P_{CC}\right)^2 &= 0. \end{aligned}$$

with $\sigma_{DD} = 1 - \sigma_{CC} - 2\sigma_{CD}$*.*

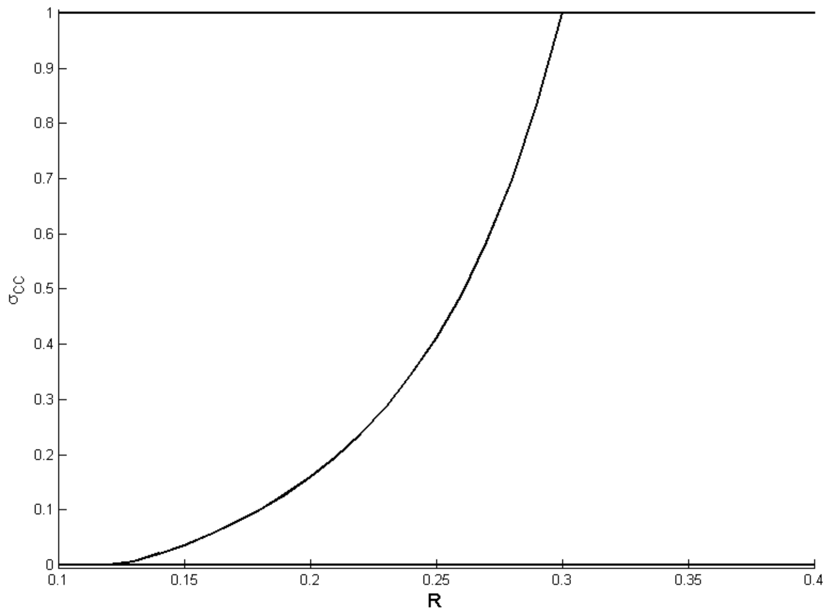
Proposition 4 states implicit conditions for the existence of interior rest points. Although it is possible to obtain explicit conditions using symbolic computer software, the number and complexity of the terms involved make any attempt to interpret these conditions futile.

Figure 3 presents the possible rest point values for σ_{CC} when the imitation function used is given by the PIR with dominant switching rate and payoffs are set to $T = 0.5$, $P = 0$, $S = -0.1$. The value of R, the payoff a cooperative couple obtain, changes from 0.1 to 0.4. We can observe that for this parametrization of the model if $R \in (0.13, 0.29)$ then, apart from the rest points $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ and $(\sigma_{CC}, \sigma_{CD}) = (1, 0)$, there also exists an interior rest point.

The behavior of the model when interior rest points exist is illustrated in figure 4. In the simulation performed on the left hand size the initial level of cooperation in relatively low, $(\sigma_{CC}, \sigma_{CD}) = (0, 0.01)$. On the contrary, in the simulation on the right hand side the initial level of cooperation in relatively high, $(\sigma_{CC}, \sigma_{CD}) = (0.99, 0)$. The parameters of the payoff matrix are set to the same values as those in figure 3 with $R = 0.25$. The imitation rule employed is again given by the PIR with dominant switching rate. The parameter values and imitation function used are such that $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ is a repeller.

As we can see on the left hand side of figure 4, the system converges to an interior rest point where about 55% of the population cooperate. On the right hand side of figure 4, the initial level of cooperation is relatively high as 99% of the population initially cooperates yet this results in the same level of cooperation in the long run as before.

Figure 3: Possible rest points values for σ_{CC} as function of R. PIR with $T = 0.5$, $P = 0$, $S = -0.1$.



When interior rest points are not present, all our local analysis in section 3.3 extends to a global level. This is due to the fact that if no interior rest point exists then the point $(\sigma_{CC}, \sigma_{DD}) = (0, 0)$ being a repeller implies that $(\sigma_{CC}, \sigma_{DD}) = (1, 0)$ is the only asymptotically stable rest point. Similarly, if $(\sigma_{CC}, \sigma_{DD}) = (0, 0)$ is asymptotically stable then $(\sigma_{CC}, \sigma_{DD}) = (1, 0)$ is a repeller.

Proposition 5. *Assume no interior rest points exist. If $f(R, P) > 2f(T, R)f(P, S)$, then for all $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$*

$$\lim_{t \to \infty} \sigma^t_{CC} = 1.$$

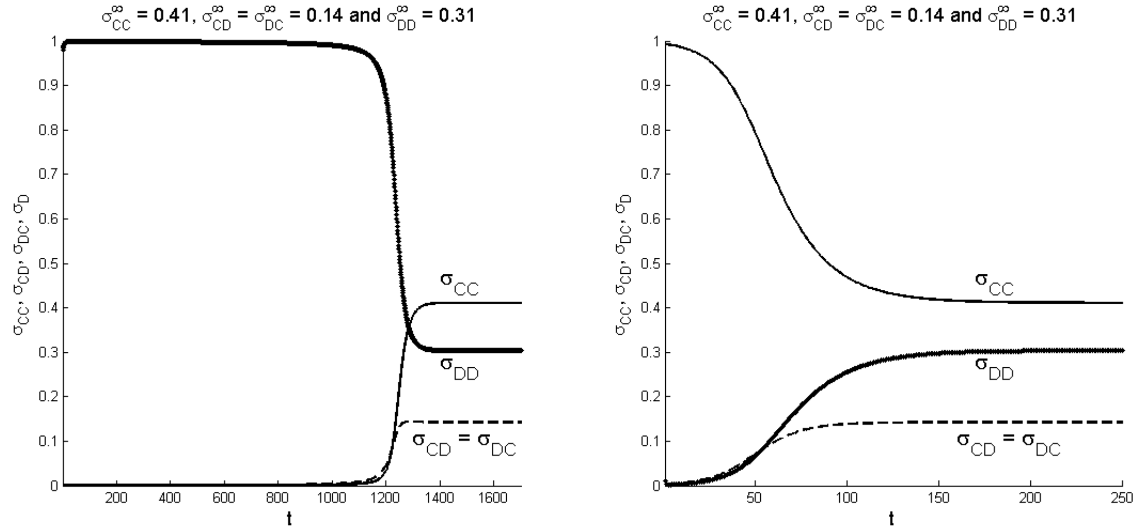
If $f(R, P) < 2f(T, R)f(P, S)$, then for all $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$

$$\lim_{t \to \infty} \sigma^t_{DD} = 1.$$

Proof. Given that no interior rest point exists and that equations (4) and (5), are continuous we have that no cycles can exist. Once this fact has been established the result of the proposition follows from proposition 3. □

Note that the analysis carried out on the PIR, IB and PRR still applies in the absence of interior rest points. What is more, results obtained for these imitation functions are strengthen as if no interior rest points are present then $(\sigma_{CC}, \sigma_{DD}) = (0, 0)$ being a repeller implies that for any $(\sigma^0_{CC}, \sigma^0_{CD}) \in \mathring{\Omega}$ all players cooperate in the long run.

Figure 4: Simulation: PIR with $T = 0.5$, $R = 0.25$, $P = 0$, $S = -0.1$. Left hand side: $(\sigma^0_{CC}, \sigma^0_{CD}) = (0, 0.01)$, right hand side: $(\sigma^0_{CC}, \sigma^0_{CD}) = (0.99, 0)$.



3.5 Cautious imitation

As already hinted when the different imitation rules where explored, we can deduce from proposition 3 that more cautions imitation makes cooperation more likely. Hence, in a setting where players are less prone to change actions cooperation has a higher chance to be present in the long run. If we consider the limit situation where players change actions very infrequently we have the following result:

Proposition 6. *For any $T > R > P > S$ with $T, R, P, S \in \mathbb{R}$ there exists a function $\bar{f}$ such that for all f with $f(x, y) < \bar{f}(x, y)$ for any $x, y \in \mathbb{R}$ the rest point $(\sigma_{CC}, \sigma_{CD}) = (0, 0)$ is a repeller.*

Proof. Take any function f that can be written as $f(\pi', \pi) = sg(\pi', \pi)$ for some $s > 0$ and some function g weakly increasing in its first argument and weakly decreasing in its second argument. Examples of such functions include the PIR, the PRR and $f(\pi', \pi) = s$.

The condition in proposition 3 means that cooperation does not vanish from the population if and only if $f(R, P) > 2f(T, R)f(P, S)$. This condition can be rewritten as $sg(R, P) > 2s^2 g(T, R)g(P, S)$. Thus, for any $T > R > P > S$ and any g we can find an $s > 0$ small enough so that cooperation survives in the long run. This gives the desired result. □

Note that proposition 6 above also means that if no interior rest points are present then for all payoff matrices we can find a function $\bar{f}$ such that for all $f < \bar{f}$ the unique stable rest point has all players cooperating.

4 Literature, Discussion and Other Matching Protocols

4.1 Literature: Bergstrom and Stark (1993) and Bergstrom (2003)

Bergstrom and Stark (1993) consider an evolutionary model where every player's behavior is hard wired to be either cooperate or defect. Each couples's offspring imitates either the behavior of their parents or the behavior of a random individual from the population and then plays a prisoner's dilemma game with each of her two siblings. The chances to survive to reproductive age depend on the payoff obtained and players that survive until reproductive age are then matched and reproduce. In Bergstrom and Stark (1993) cooperation does not survive if the offspring always imitate a random player from the population. This is the case as in their model the prisoner's dilemma game is played with one's siblings, whose behavior may not represent the average behavior in the population. That is, the prisoner's dilemma game is played locally with one's siblings yet the imitation takes place at a population level. In our model, imitation also takes place at a population level, all players in the population are equally likely to be observed. However, players who are not in a cooperative couple are randomly matched with another player from the entire population of non-cooperative couples. That is, as opposite to our model, in Bergstrom and Stark (1993) matching is local.

Bergstrom (2003) presents an evolutionary model where, as in Bergstrom and Stark (1993), players are hard wired to be either cooperators or defectors. In Bergstrom (2003) the shares of each of the two types of players in the population change according to their expected payoff. Thus, for instance, if cooperators get higher payoff than defectors then their share in the population increases whilst the share of defectors in the population decreases. In Bergstrom (2003) matching is assortative as the probability of meeting a player of the same type is different than the probability of meeting a player of a different type. The author shows conditions in the probability that matchings are assortative under which cooperation prevails in the long run. As we discuss below, if matching is completely assortative: cooperators only meet cooperators and defectors only meet defectors, then cooperation is more likely to arise than in the model presented in this paper, where assortative matching only occurs when the two players in a pair cooperate.

4.2 Discussion

The long run behavior of the population can be determined to a certain extend by the initial condition. For example, if no player cooperates initially, then no player ever cooperates. This fact disappears if, for example, mutations or mistakes are introduced in the model. Given that we are dealing with a continuum of population, introducing mistakes is straightforward.

Assume that at any given period with a small probability $\varepsilon > 0$ each player makes a mistake and chooses the action she intended not to. In this case and given that an infinite of population exists, each period exactly a fraction ε of players make mistakes. More specifically, a fraction $\varepsilon(\sigma_{CC} + \sigma_{CD})$ of players that intended to choose C play D, and a fraction $\varepsilon(\sigma_{DC} + \sigma_{DD}$ of players that intended to choose D play C.

Results presented are still valid if, in the model with mistakes, a rest point is defined as the situation where for any ε the change in σ_{CC} and σ_{CD} is always smaller or equal than $\varepsilon\sigma_{CC}$ and $\varepsilon\sigma_{CD}$ respectively. The convenience of adding mistakes is that unstable rest points are eliminated. That is, in the model with mistakes, if $f(R, P) > 2f(T, R)f(P, S)$, then cooperation emerges independently of the initial conditions.

In the model presented, when it comes to imitating another player all agents in the population are equally likely to be observed. A sensible alternative is then to have correlation in sampling. For instance, one can consider a situation where cooperators are more likely to observe other cooperators and defectors are more likely to observe other non-cooperative players. In this case, if cooperation can be present in the model considered in this paper then cooperation is more likely in a setting where there is correlation in sampling. This is the case as if players are more prone to observe those who choose their same action, then chances of imitating are lower as a requirement for imitation is that a player choosing a different action should be observed. However, as it can be inferred from proposition 6, if the probability of imitating is lower, then cooperation is more likely to be present in the long run.

4.3 Other Matching Protocols

In this paper, we consider a matching mechanism whereby only the pairs where both players cooperate are maintained. This matching mechanism captures the simple idea that a player should have no incentives to repeat partner unless the partner played cooperatively last period. There are, however, other matching settings that could be considered. In this subsection we explore different matching protocols as well as justify why players may have incentives to keep cooperative partners only.

Assortative matching

An alternative matching protocol is such that all players who choose the same strategy as their partners are rematched. In this case cooperation is possible for a bigger set of payoff matrices and imitation functions. This is the case since the payoff of non-cooperative players is lower than in our original model as this players are less likely to be matched with a player being cooperative. In the model presented, this is as if players in σ_{DD} repeat partner and, thus, cannot be matched with a player in σ_{CD}, which is the matching that gives the highest payoff to defectors.

All pairs are kept

Another sensible option is to assume that players always keep their partners. In this case, cooperation is possible for a bigger set of payoff matrices and imitation functions as defectors are less likely to find a cooperator to take advantage of. That is, if all players repeat partner, then assortative matching tends to occur faster. This is the case as the cooperative players that are matched with a non-cooperative one are more likely to change to the non-cooperative action as they repeat couple and their partner does not change action (as she gets the highest possible payoff). However, a population that is separated between cooperators and defectors is not stable as the former always get more payoff than the latter. Thus, the share of cooperative pairs increases as gradually every two players in a non-cooperative pair switch simultaneously to the cooperative action.

Pairs are kept with some fixed probability

A further matching protocol is such that players keep their partner with some exogenous probability. This setting is a mixture between the case where players never keep their partner (random matching, section 3.1) and the case just described above. Therefore, the chances with which cooperation survives depend on the exogenous probability by which pairs are kept on top of the payoff matrix and the specific imitation function.

Correlation is not perfect

A fourth alternative has cooperative pairs maintained with a probability that is less than one. This imperfect correlation setting shrinks the set of payoff matrices and imitation functions for which cooperation survives as what makes cooperation possible in the main model are the benefits from repeating partner when both parties cooperate.

Why keeping only cooperative partners?

We argued in the introduction that it seems a reasonable rule of thumb not to keep a non-cooperative partner. A question is then to which extend this rule of thumb can appear if players rationally decide whether to keep their partner or not. Given that players prefer cooperative partners simply because facing a cooperative player strictly payoff dominates facing a non-cooperative one, a rational player chooses the option where the chances of finding a cooperative partner are highest.

If players in σ_{CC} keep their current couple then the chances of having a cooperative partner are $1 - P_{CC}$. If, however, they choose not to keep their couple, then they are matched with another player at random from the population of σ_{CD}, σ_{DC} and σ_{DD}. In this case the chances of finding a cooperative partner are $\frac{1}{1-\sigma_{CC}}\left(\sigma_{CD}(1-P_{CD})+\sigma_{DD}P_{DD}\right)$. It is not hard to show that

$$1 - P_{CC} > \frac{1}{1-\sigma_{CC}}\left(\sigma_{CD}(1-P_{CD})+\sigma_{DD}P_{DD}\right)$$

and, thus, chances of having a cooperative couple are highest for players in σ_{CC} if they keep their current partner.

Players in σ_{CD} want to change partner because their couple plays D again as she obtained the maximum possible payoff. On the other hand, by changing partner there is some positive probability of facing a cooperative partner.

The decision of players in σ_{DC} is irrelevant given that, as we just argued, their couple always wants to change partner.

Finally, players in σ_{DD} have incentive to change partners if their chances of meeting a cooperative player increase by facing a random partner. That chances that their current partner cooperates is given by $P_{DD} = \sigma_{CC} f(R, P)$ whilst the chances of having a cooperative partner if they change their couple are given by $\frac{1}{1-\sigma_{CC}}\left(\sigma_{CD}(1-P_{CD})+\sigma_{DD}P_{DD}\right)$. It is not hard to show that if $f(R, P) \leq 0.5$ then

$$P_{DD} < \frac{1}{1-\sigma_{CC}}\left(\sigma_{CD}(1-P_{CD})+\sigma_{DD}P_{DD}\right)$$

and, thus, all players in σ_{DD} have the highest chances of meeting a cooperative player if they change their partner. If $f(R, P) > 0.5$ then it is possible that players in σ_{DD} prefer to keep their current partner. As already argued above when assortative matching was considered, if this was allowed then cooperation is possible for a bigger set of payoff matrices and imitation functions.

5 Conclusions

The present paper investigated cooperation in a setting where players who learn by imitation are matched to play a Prisoner's Dilemma game. Our contribution to the literature lies in the way matching takes place: players that belong to a pair were both parties cooperated repeat partner while the rest of players are randomly matched into pairs.

Our analysis has shown that the way interactions take place in an imitative population have a crucial effect on whether cooperation can be sustained in a society. Imitative behavior is based on the selfish consideration of copying successful actions taken by others. However, when we introduce certain rematching in the way interactions take place, this selfish behavior can lead to a cooperative society.

Amongst other findings, we have given a precise answer to when cooperation does not disappear in a population. Nevertheless, it may be possible to obtain further results by simplifying the model or by taking a different modelling approach. We leave this tasks for possible future research.

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Appendix

Random Matching

With random matching, there is no need to distinguish between players who cooperated and were paired with a player who also cooperated, σ_{CC}, and players who cooperated and faced a player who did not cooperate, σ_{CD}. Thus, these two sets of players are grouped into the same set σ_C. Similarly, the sets σ_{DC} and σ_{DD} are grouped into the set σ_D.

Let σ_C^{t+1} the fraction of players who chose C at time t with $\sigma_C^0 \in [0,1]$ given. Let $1-\sigma_C$ be the fraction of players who chose D at time t. Furthermore, let $P_C : A^2 \times \mathbb{R}^2 \to [0,1]$ be the probability with which a cooperative player switches to D and let $P_D : A^2 \times \mathbb{R}^2 \to [0,1]$ be the probability with which a player who chose D switches to C. Assume P_C and P_D satisfy Assumptions $1-3$. The evolution of σ_C is then given by

$$\sigma_C^{t+1} \quad = \quad \sigma_C^t(1-P_C) + (1-\sigma_C)P_D. \tag{8}$$

Proceeding in a similar fashion as in equation (1), P_C is positive only if the player in σ_C observed a player in $1-\sigma_C$. Three different situations can occur now: First, if the player in σ_C faced a player in σ_C and observed a player who faced another one in σ_C, then own payoff equals R while observed payoff equals T. Second, if the player in σ_C faced a player in $1-\sigma_C$ and observed a player who faced another one in σ_C, then own payoff equals S while observed payoff equals T. Finally, if the player in σ_C faced a player in $1-\sigma_C$ and observed a player who faced another one in $1-\sigma_C$, then own payoff equals S while observed payoff equals P. Therefore, we have that

$$P_C \quad = \quad (1-\sigma_C)\left[\sigma_C^2 f(T,R) + (1-\sigma_C)\sigma_C f(T,S) + (1-\sigma_C)^2 f(P,S)\right]. \tag{9}$$

On the other hand, we have that P_D is positive only if the player in $1-\sigma_C$ faced another one who played D, and observes an individual choosing C that faced a player who also chose

C. In this case, observed payoff equals R while own payoff equals P. Hence, we can write P_D as follows:

$$P_D \quad = \quad (1-\sigma_C)\sigma_C^2 f(R,P). \tag{10}$$

Lemma 1. *With random matching only $\sigma_C = 1$ and $\sigma_C = 0$ are rest points. Furthermore, for any $\sigma_C^0 \in (0,1)$*

$$\lim_{t\to\infty} \sigma_C^t \quad = \quad 0.$$

Proof. We can see from equation (8) that both $\sigma_C = 1$ and $\sigma_C = 0$ are rest points. The proof is completed by showing that from any point $\sigma_C \in (0,1)$ the system converges to $\sigma_C = 0$.

If $\sigma_C \in (0,1)$, using Assumptions 2 and 3 we obtain the following:

$$\begin{aligned}
\sigma_C^2 f(T,R) + (1-\sigma_C)\sigma_C f(T,S) + (1-\sigma_C)^2 f(P,S) \quad &> \quad (1-\sigma_C)\sigma_C f(T,S) \\
&\geq \quad (1-\sigma_C)\sigma_C f(T,P) \\
&\geq \quad (1-\sigma_C)\sigma_C f(R,P).
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\sigma_C f(R,P) \quad < \quad & \sigma_C^2 f(R,P) + \sigma_C^2 f(T,R) + \\
& (1-\sigma_C)\sigma_C f(T,S) + (1-\sigma_C)^2 f(P,S).
\end{aligned}$$

Multiply both sides by $\sigma_C(1-\sigma_C)$ and use equations (10) and (9) to obtain

$$P_D \quad < \quad \sigma_C \left(P_C + P_D\right). \tag{11}$$

From (8) we have that $\Delta\sigma_C = P_D - \sigma_C(P_C + P_D)$. Hence, by equation (11), we know that whenever $\sigma_C \in (0,1)$, $\Delta\sigma_C < 0$. Thus, no point $\sigma_C \in (0,1)$ can be a rest point and the system cannot converge to $\sigma_C = 1$ from any initial condition $\sigma_C \in (0,1)$.

We still have to show that the system cannot converge to a point that is not a rest point. This is straightforward since $\Delta\sigma_C$ is a polynomial in σ_C and, hence, continuous for all $\sigma_C \in [0,1]$. □

Proof of proposition 4

Proof. Necessary and sufficient conditions for $(\sigma_{CC}, \sigma_{CD}) \in \mathring{\Omega}$ to be a rest point are that neither the total share of cooperators nor the fraction of players in σ_{CC} change over time. That is, necessary and sufficient conditions are

$$\begin{aligned}
\sigma_{CC}^{t+1} + \sigma_{CD}^{t+1} \quad &= \quad \sigma_{CC}^t + \sigma_{CD}^t, \\
\sigma_{CC}^{t+1} \quad &= \quad \sigma_{CC}^t.
\end{aligned}$$

Thus, using equations (4) and (5) we have that

$$\begin{aligned}\sigma_{CC}P_{CC}+\sigma_{CD}P_{CD}-\sigma_{DD}P_{DD} &= 0,\\ \sigma_{CC}P_{CC}\left(2-P_{CC}\right)\left(1-\sigma_{CC}\right)-\left(\sigma_{CD}\left(1-P_{CD}\right)+\sigma_{DD}P_{DD}\right)^2 &= 0.\end{aligned}$$

The two equations above can be rewritten as

$$\begin{aligned}\sigma_{CC}P_{CC}+\sigma_{CD}P_{CD}-\sigma_{DD}P_{DD} &= 0,\\ \sigma_{CC}P_{CC}\left(2-P_{CC}\right)\left(1-\sigma_{CC}\right)-\left(\sigma_{CD}+\sigma_{CC}P_{CC}\right)^2 &= 0.\end{aligned}$$

Which concludes the proof. □

Proof of proposition 5

Proof. Define the set $\Sigma_r=(\sigma_{CC},\sigma_{CD})\in\mathring{\Omega}\cap B_r(0,0)$. For sufficiently small $\varepsilon>0$ we can disregard terms of order $o(\varepsilon^2)$ and write the system (4) and (5) when $(\sigma_{CC},\sigma_{CD})\in\Sigma_\varepsilon$ as

$$\begin{aligned}\sigma_{CC}^{t+1}-\sigma_{CC}^{t} &= 0,\\ \sigma_{CD}^{t+1}-\sigma_{CD}^{t} &= \sigma_{DD}^{t}\left(f(R,P)\sigma_{CC}^{t}-f(P,S)\sigma_{CD}^{t}\right).\end{aligned}$$

The approximation above is correct up to a term of order ε^2. Thus, when the process is arbitrarily close to $(0,0)$, the change in σ_{CC} with respect to the change in σ_{CD} is negligible.

The system above converges to $\sigma_{CD}=\sigma_{CC}\frac{f(R,P)}{f(P,S)}$[5]. Hence, if we start in Σ_ε with ε small, the process converges to a situation where $\sigma_{CD}=\sigma_{CC}\frac{f(R,P)}{f(P,S)}$. The system may hit the path $\sigma_{CD}=\sigma_{CC}\frac{f(R,P)}{f(P,S)}$ outside the set Σ_ε. This poses no problem as the further away from $(0,0)$ the system can be in this case is within the set $\Sigma_{\varepsilon\frac{f(R,P)}{f(P,S)}}$, which is also arbitrarily close to $(0,0)$ when ε is small.

After starting in Σ_ε and once the system reaches $\sigma_{CD}=\sigma_{CC}\frac{f(R,P)}{f(P,S)}$, we can rewrite (4) as

$$\sigma_{CC}^{t+1}-\sigma_{CC}^{t} = \left(\sigma_{CC}^{t}\right)^2\frac{f(R,P)}{f(P,S)}\left(\frac{f(R,P)}{f(P,S)}-2f(T,R)\right).$$

The equation of the motion of σ_{CD} is irrelevant because in the neighborhood of $(0,0)$ the system moves along the path $\sigma_{CD}=\sigma_{CC}\frac{f(R,P)}{f(P,S)}$ as we just proved. To be more precise, the Center Manifold Theorem is being used here (see Sastry (1999) Section 7.8 or Khalil (1995) Section 8.1).

By Bézout's Theorem, the system (5) and (4) has a finite number of solutions (see Kirwan (1992)). Thus, we can fix $\varepsilon>0$ such that no rest point points exists in $\Sigma_\varepsilon\setminus(0,0)$.

[5]If $f(P,S)\neq 0$ then the result in the lemma follows.

For any $\kappa < \varepsilon$, if $f(R,P) > 2f(T,R)f(P,S)$ then $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$. Thus, since $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$ and $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$, if the system starts in the boundary of Σ_κ, then it will leave that set. Assume that the system, after leaving Σ_κ, does not hit the boundary of the other bigger set Σ_ε. Since for any point in Σ_ε we have that $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$, by continuity of (4) and (5) if the process does not hit the boundary of Σ_ε then we must have that there exists a point $(\sigma_{CC}, \sigma_{CD}) \in \Sigma_\varepsilon \smallsetminus (0,0)$ such that $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} = 0$ and, thus, $\sigma_{CD}^{t+1} - \sigma_{CD}^{t} = 0$. That is, there must exists at least one rest point point in $\Sigma_\varepsilon \smallsetminus (0,0)$, which is a contradiction.

Thus, if the process starts in Σ_κ, then it must hit the boundary of Σ_ε. We know that for any point in Σ_ε, if $f(R,P) > 2f(T,R)f(P,S)$ then $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} > 0$ and $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$. Thus, starting in boundary of Σ_κ the process leaves Σ_ε, which is the condition for the point $(0,0) \in \Omega$ to be a repeller.

Assume now that $f(R,P) < 2f(T,R)f(P,S)$. By continuity, $\sigma_{CC}^{t+1} - \sigma_{CC}^{t} < 0$, $\sigma_{CD} = \sigma_{CC}\frac{f(R,P)}{f(P,S)}$ and the fact that no rest point point exists in $\Sigma_\varepsilon \smallsetminus (0,0)$, if the system starts in $\Sigma_\varepsilon \smallsetminus \Sigma_\kappa$ then it eventually enters the set Σ_κ for any $\kappa < \varepsilon$. This is the condition for asymptotic stability. □