

# **Polarization and coherence for vectorial electromagnetic waves and the ray picture of light propagation**

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## **Abstract**

We develop a complete geometrical picture of paraxial light propagation including coherence phenomena. This approach applies both for scalar and vectorial waves via the introduction of a suitable Wigner function and can be formulated in terms of an inverted Huygens principle. Coherence is included by allowing the geometrical rays to transport generalized Stokes parameters. The degree of coherence for scalar and vectorial light can be expressed as simple functions of the corresponding Wigner function.

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## I. INTRODUCTION

In this work we develop a complete geometrical formulation of paraxial optics for vectorial electromagnetic waves. By geometrical we mean that this propagation is fully described in terms of the light rays of geometrical optics. By complete we mean that includes without approximation all coherence phenomena. This is equivalent to define a Wigner function for vectorial waves, which in turn is equivalent to the prescription of a set of Stokes parameters to geometrical rays [1–4].

This task is interesting since it merges in a single formalism geometrical and wave optics. This may provide physical insight and simple formulas for many problems where coherence is involved such as image formation with polarized light.

We will apply this formulation to the degree of coherence for vectorial waves. This is a fundamental nontrivial problem only recently addressed in depth. Contrary to the scalar case, for vectorial waves there is no unique degree of coherence, and currently several definitions coexist [5–22]. After recalling the main proposals we examine their relationship with the geometrical picture.

## II. WIGNER FUNCTION FOR SCALAR WAVES

We first recall the geometrical Wigner formulation of paraxial optics for scalar waves. Although standard geometrical optics excludes coherent phenomena, if we replace ray intensity by Wigner function we include once for all coherence effects. In particular we can express the degree of coherence as a functional of the Wigner function. The price to be paid is that the Wigner function can take negative values so it cannot represent always light intensity.

### A. Definition and properties

We will always consider the spatial-frequency domain so that the Wigner function is defined in terms of the cross-spectral density function as [23–27]

$$W(\mathbf{r}, \mathbf{p}) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} d^2\mathbf{r}' \langle E(\mathbf{r} - \mathbf{r}'/2)E^*(\mathbf{r} + \mathbf{r}'/2) \rangle \exp(ik\mathbf{p} \cdot \mathbf{r}'), \quad (1)$$

where the angle brackets represent ensemble average,  $\mathbf{r}$  and  $\mathbf{r}'$  are Cartesian coordinates in a plane orthogonal to the main propagation direction along axis  $z$ ,  $\mathbf{p}$  are the angular variables

representing the local direction of propagation, and  $k$  is the wavenumber in vacuum.

The connection between Wigner function and geometrical optics stems from the fact that  $\mathbf{r}$  and  $\mathbf{p}$  represent the parameters of a light ray, so that  $W$  assigns a number to each ray. The main properties of this formalism are:

(a) The Wigner function provides complete information about second-order phenomena, including diffraction and interference, since its definition can be inverted to express the cross-spectral density function in terms of the Wigner function

$$\langle E(\mathbf{r}_1)E^*(\mathbf{r}_2) \rangle = \int_{-\infty}^{\infty} d^2\mathbf{p}W(\mathbf{R}, \mathbf{p}) \exp [ik\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)], \quad (2)$$

where  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$  is the midpoint between  $\mathbf{r}_{1,2}$ .

(b) In particular, the light intensity (irradiance) at a given point can be obtained by integrating the angular variables

$$I(\mathbf{r}) = \langle |E(\mathbf{r})|^2 \rangle = \int d^2\mathbf{p}W(\mathbf{r}, \mathbf{p}). \quad (3)$$

This is to say that the intensity at a given point is the sum of the values of the Wigner function for all the rays passing through this point. We will refer to this sum as an incoherent superposition since the contributions of all rays are independent.

(c) The Wigner function cannot represent always light intensity since it can take negative as well as positive values. We may say that there are bright rays with positive  $W$  and dark rays with negative  $W$ . Dark rays are crucial to the completeness of the theory since they contain the coherence as we shall see below.

(d) Finally a crucial property for the geometrical interpretation of the Wigner function is that it is constant along paraxial rays

$$W^{(z)}(\mathbf{r}', \mathbf{p}') = W^{(0)}(\mathbf{r}, \mathbf{p}), \quad (4)$$

where  $(\mathbf{r}, \mathbf{p})$  and  $(\mathbf{r}', \mathbf{p}')$  are the input (plane  $z = 0$ ) and output (plane  $z$ ) ray parameters through a paraxial optical system.

## B. Inverted Huygens principle

These properties can be summarized in a principle analogous to the Huygens principle but with inverted terms replacing waves by rays and coherent by incoherent superpositions. We can enunciate this principle in three steps [28]:

(i) Each point acts as a secondary source of a continuous distribution of rays with parameters  $\mathbf{r}, \mathbf{p}, W(\mathbf{r}, \mathbf{p})$ . We stress that this is a continuous distribution of rays instead of the more familiar single ray at each point normal to a wavefront.

(ii) The evolution of optical properties is given by the incoherent superposition of rays, as illustrated by the example of light intensity. We stress that this incoherence is a key feature of the theory independent of the actual state of coherence of the light. Coherence is expressed in a different way as we shall see later on.

(iii) The effect of spatial local filters (transparencies) altering phase and amplitude is described in the wave picture by the product of the amplitude of the input wave with the transmission coefficient  $t(\mathbf{r})$ , i. e.,  $U(\mathbf{r}) \rightarrow t(\mathbf{r})U(\mathbf{r})$ . In the geometrical picture this effect is described by the convolution product of the input Wigner function  $W_U$  with the Wigner function  $W_t$  of the transmission coefficient  $W_U \rightarrow W_{tU} = W_t \star W_U$

$$W_{tU}(\mathbf{r}, \mathbf{p}) = \int d^2\mathbf{p}' W_t(\mathbf{r}, \mathbf{p} - \mathbf{p}') W_U(\mathbf{r}, \mathbf{p}'), \quad (5)$$

where

$$W_t(\mathbf{r}, \mathbf{p}) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} d^2\mathbf{r}' t(\mathbf{r} - \mathbf{r}'/2) t^*(\mathbf{r} + \mathbf{r}'/2) \exp(ik\mathbf{p} \cdot \mathbf{r}'). \quad (6)$$

### C. Coherence

The degree of coherence for scalar waves can be expressed in terms of the Wigner function from two different perspectives.

#### 1. Coherence as phase-difference average

On the one hand, from the inversion formula (2) expressing the cross-spectral density function in terms of the Wigner function, we have that the degree of coherence at two points  $\mathbf{r}_{1,2}$  is the average of the phase difference  $\varphi = k\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)$  at  $\mathbf{r}_{1,2}$  produced by an ensemble of plane waves with wavevectors proportional to  $\mathbf{p}$

$$\mu = \frac{|\langle E(\mathbf{r}_1)E^*(\mathbf{r}_2) \rangle|}{\sqrt{I_1 I_2}} = \frac{|\langle \exp(i\varphi) \rangle_W|}{\sqrt{I_1 I_2}}, \quad (7)$$

where the weight of each plane wave is  $W(\mathbf{r}, \mathbf{p})$  with  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ , and  $I_{1,2}$  are the light intensities at points  $\mathbf{r}_{1,2}$  [28]. This agrees well with common intuition since rays are usually

understood as local plane waves, and partial coherence is usually understood as the result of phase fluctuations.

## 2. Coherence in a Young interferometer

On the other hand, we can examine the ray picture of coherence in action, for example in the Young interferometer with two apertures at points  $\mathbf{r}_{1,2}$ . The Wigner function after the apertures implies the existence of just three secondary sources of rays [22]. Two real sources located at the apertures that emit isotropically bright rays, with Wigner function proportional to the field intensity  $W(\mathbf{r}_{1,2}, \mathbf{p}) = W(\mathbf{r}_{1,2}) \propto I_{1,2}$  independent of  $\mathbf{p}$ , and a fictitious source of bright and dark rays at the midpoint between the apertures  $W(\mathbf{R}, \mathbf{p})$ . Dark and bright rays are in an equal number since

$$\int_{\mathbf{R}} d^2\mathbf{r} d^2\mathbf{p} W(\mathbf{r}, \mathbf{p}) = 0, \quad (8)$$

because of the fictitious character of the source.

Each observation point  $\mathbf{r}$  in an interference plane at a distance  $z$  from the plane of the apertures is reached exclusively by three rays, one from each source. Their incoherent superposition gives the intensity distribution

$$I(\mathbf{r}) \propto W(\mathbf{r}_1) + W(\mathbf{r}_2) + W(\mathbf{R}, \mathbf{p}). \quad (9)$$

The contribution  $W(\mathbf{R}, \mathbf{p})$  from the midpoint is actually the interference term since it is the only one that depends on the observation point through the direction  $\mathbf{p}$

$$\mathbf{p} = n \frac{\mathbf{r} - \mathbf{R}}{z}, \quad (10)$$

which is the propagation direction followed by the ray from the source to the observation point.

This implies a close relation between the degree of coherence and the Wigner function in the midpoint and also with the negativity of the Wigner function. Such a relation can be expressed by three equations [28, 29]:

(1) The interference term is of the form

$$W(\mathbf{R}, \mathbf{p}) = 2\mu \sqrt{W(\mathbf{r}_1)W(\mathbf{r}_2)} \cos [k\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2) + \delta], \quad (11)$$

where  $\delta$  is a constant phase, so  $\mu$  is proportional to the maximum modulus of the Wigner function at the midpoint  $|W(\mathbf{R}, \mathbf{p})|$  when  $\mathbf{p}$  is varied

$$\mu = \frac{|W(\mathbf{R}, \mathbf{p})|_{\max}}{2\sqrt{W(\mathbf{r}_1)W(\mathbf{r}_2)}}. \quad (12)$$

(2) The degree of coherence is proportional to the negativity of the Wigner function measured as the distance of  $W$  to its modulus

$$\mu^2 = \frac{k^2 \int d^2\mathbf{r} d^2\mathbf{p} [W(\mathbf{r}, \mathbf{p}) - |W(\mathbf{r}, \mathbf{p})|]^2}{2\pi^2 W(\mathbf{r}_1)W(\mathbf{r}_2)}. \quad (13)$$

(3) The degree of coherence is proportional to the amount of Wigner function in the midpoint measured as

$$\mu^2 = \frac{2k^2 \int_R d^2\mathbf{r} d^2\mathbf{p} W^2(\mathbf{r}, \mathbf{p})}{\pi^2 W(\mathbf{r}_1)W(\mathbf{r}_2)}, \quad (14)$$

where the integration extends just to the region  $R$  between apertures.

Note that from this interferometric point of view coherence is incompatible with standard geometrical optics represented by always positive  $W$ . In other words, coherence between two points is the distance of the light state after the apertures to standard geometrical optics.

### III. WIGNER FUNCTION FOR VECTORIAL WAVES

#### A. Definition and properties

The Wigner function we are going to use is the translation to optics of a similar Wigner function introduced in mechanics to describe a closely related problem, the Wigner function of a particle with spin one half [30–32]. This is equivalent to a transversal wave since in both cases we have a field with two components. Such a Wigner function can be expressed in optics as [1]

$$W(\mathbf{r}, \mathbf{p}, O) = \mathbf{S}(\mathbf{r}, \mathbf{p}) \cdot \mathbf{O}, \quad (15)$$

where  $\mathbf{O}$  is a four-dimensional real vector that represents the Poincaré sphere,

$$\mathbf{O} = \frac{1}{2} \left( 1, \sqrt{3} \sin \theta \cos \phi, \sqrt{3} \sin \theta \sin \phi, \sqrt{3} \cos \theta \right), \quad (16)$$

and  $\mathbf{S}(\mathbf{r}, \mathbf{p})$  is a four-dimensional real vector with components

$$S_0(\mathbf{r}, \mathbf{p}) = W_{x,x}(\mathbf{r}, \mathbf{p}) + W_{y,y}(\mathbf{r}, \mathbf{p}),$$

$$\begin{aligned}
S_1(\mathbf{r}, \mathbf{p}) &= W_{x,y}(\mathbf{r}, \mathbf{p}) + W_{y,x}(\mathbf{r}, \mathbf{p}), \\
S_2(\mathbf{r}, \mathbf{p}) &= i [W_{x,y}(\mathbf{r}, \mathbf{p}) - W_{y,x}(\mathbf{r}, \mathbf{p})], \\
S_3(\mathbf{r}, \mathbf{p}) &= W_{x,x}(\mathbf{r}, \mathbf{p}) - W_{y,y}(\mathbf{r}, \mathbf{p}),
\end{aligned} \tag{17}$$

where  $W_{\cdot,m}$  are the elements of the Wigner matrix

$$W_{\cdot,m}(\mathbf{r}, \mathbf{p}) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} d^2\mathbf{r}' \langle E(\mathbf{r} - \mathbf{r}'/2) E_m^*(\mathbf{r} + \mathbf{r}'/2) \rangle \exp(ik\mathbf{p} \cdot \mathbf{r}'). \tag{18}$$

We can appreciate that the Wigner function depends on the spherical coordinates  $O$  that represents the information about the polarization state. We can appreciate also that the Stokes parameters are ray properties because of their joint dependence on  $(\mathbf{r}, \mathbf{p})$ .

The properties of this Wigner function are fully equivalent to the scalar case [1–3]:

(a) The Wigner function provides complete information about second-order phenomena, since its definition can be inverted

$$\langle E(\mathbf{r}_1) E_m^*(\mathbf{r}_2) \rangle = \frac{1}{2} \int_{-\infty}^{\infty} d^2\mathbf{p} \sum_{j=0}^3 S_j(\mathbf{R}, \mathbf{p}) \sigma_{\cdot,m}^{(j)} \exp(ik\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)), \tag{19}$$

where  $\sigma^{(j)}$  are the Pauli matrices,  $\sigma^{(0)}$  being the identity, and  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ .

(b) In particular, at each spatial point  $\mathbf{r}$  the intensity and the polarization state, represented by the point Stokes parameters  $s_0(\mathbf{r})$  and  $s_{1,2,3}(\mathbf{r})$ , respectively, are obtained from the ray Stokes parameters by integrating the angular variables

$$\mathbf{s}(\mathbf{r}) = \int d^2\mathbf{p} \mathbf{S}(\mathbf{r}, \mathbf{p}). \tag{20}$$

This is the incoherent superposition of the ray Stokes parameters associated to all the rays passing through point  $\mathbf{r}$ .

(c) The Wigner matrix may have negative eigenvalues so that the ray Stokes parameters may violate the ray analog of the relation always satisfied by the point Stokes parameters  $s_0 \geq \sqrt{s_1^2 + s_2^2 + s_3^2} \geq 0$ . The rays satisfying  $S_0 \geq \sqrt{S_1^2 + S_2^2 + S_3^2} \geq 0$  may be called bright rays, while the other ones, i. e.,  $S_0 < \sqrt{S_1^2 + S_2^2 + S_3^2}$  may be called dark rays.

(d) Finally a crucial property for the geometrical interpretation of the Wigner function is that it is constant along paraxial rays, except for the action of homogeneous polarization changing devices described by a Mueller matrix  $\mathbf{M}$

$$\mathbf{S}^{(z)}(\mathbf{r}', \mathbf{p}') = \mathbf{M} \mathbf{S}^{(0)}(\mathbf{r}, \mathbf{p}), \tag{21}$$

where  $(\mathbf{r}, \mathbf{p})$  and  $(\mathbf{r}', \mathbf{p}')$  are the ray parameters at the input ( $z = 0$ ) and output ( $z > 0$ ) planes of a paraxial optical system.

## B. Inverted Huygens principle for vectorial light

These properties can be summarized in a principle analogous to the Huygens principle but with inverted terms, by replacing waves by rays and coherent by incoherent superpositions [33]:

(i) Each point acts as a secondary source of a continuous distribution of rays with parameters  $\mathbf{r}, \mathbf{p}, \mathbf{S}(\mathbf{r}, \mathbf{p})$ .

(ii) These rays are superimposed incoherently, as illustrated by the example of point intensity and polarization.

(iii) Spatial-local inhomogeneous filters altering phase and amplitude (i. e. transparencies) are described in the wave picture by the product with transmission coefficients

$$\langle E(\mathbf{r}_1)E_m^*(\mathbf{r}_2) \rangle \rightarrow \sum_{j,k} t_{j,k}(\mathbf{r}_1) \langle E_j(\mathbf{r}_1)E_k^*(\mathbf{r}_2) \rangle t_{m,k}^*(\mathbf{r}_2), \quad (22)$$

where  $t_{j,k}(\mathbf{r})$  are the corresponding transmission coefficients. In the geometrical picture these devices are described by the convolution of the input ray Stokes parameters  $\mathbf{S}$  with the Wigner function of the Mueller matrix

$$\mathbf{S}'(\mathbf{r}, \mathbf{p}) = \int d^2\mathbf{p}' \mathbf{M}_W(\mathbf{r}, \mathbf{p} - \mathbf{p}') \mathbf{S}(\mathbf{r}, \mathbf{p}'), \quad (23)$$

where

$$\mathbf{M}_W(\mathbf{r}, \mathbf{p}) = \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} d^2\mathbf{r}' \mathbf{M}(\mathbf{r} - \mathbf{r}'/2, \mathbf{r} + \mathbf{r}'/2) \exp(ik\mathbf{p} \cdot \mathbf{r}'). \quad (24)$$

## IV. DEGREE OF COHERENCE FOR VECTORIAL WAVES

The proper definition of the degree of coherence for vectorial waves is a nontrivial problem. The increase of the number of degrees of freedom implies that an scalar quantity (the cross-spectral density function) is replaced by a matrix (the cross-spectral density matrix)

$$\mu \propto \langle E(\mathbf{r}_1)E^*(\mathbf{r}_2) \rangle \rightarrow \Gamma_{1,2} = \begin{pmatrix} \langle E_x(\mathbf{r}_1)E_x^*(\mathbf{r}_2) \rangle & \langle E_x(\mathbf{r}_1)E_y^*(\mathbf{r}_2) \rangle \\ \langle E_y(\mathbf{r}_1)E_x^*(\mathbf{r}_2) \rangle & \langle E_y(\mathbf{r}_1)E_y^*(\mathbf{r}_2) \rangle \end{pmatrix}, \quad (25)$$

so there is no straightforward translation of  $\mu$  from the scalar to the vectorial case. From the same reasons, several definitions can coexist since they will focus on different features of coherence with application to different situations. Here we can recall the main approaches to the problem.

### A. Intensity fringes in a Young interferometer

A first approach to the degree of coherence at two points  $\mathbf{r}_{1,2}$  is derived directly in terms of the visibility of interference fringes in a Young interferometer with apertures at  $\mathbf{r}_{1,2}$  where only the intensity is measured in the observation plane, leading to [5–10]

$$\mu_1^2 = \frac{|\langle E_x(\mathbf{r}_1)E_x^*(\mathbf{r}_2) \rangle + \langle E_y(\mathbf{r}_1)E_y^*(\mathbf{r}_2) \rangle|^2}{I(\mathbf{r}_1)I(\mathbf{r}_2)}, \quad (26)$$

or, equivalently,

$$\mu_1^2 = \frac{|\text{tr}\Gamma_{1,2}|^2}{\text{tr}\Gamma_{1,1}\text{tr}\Gamma_{2,2}} = \frac{|\langle \mathbf{E}(\mathbf{r}_1) \cdot \mathbf{E}^*(\mathbf{r}_2) \rangle|^2}{I(\mathbf{r}_1)I(\mathbf{r}_2)}, \quad (27)$$

where

$$\Gamma_{j,j} = \begin{pmatrix} \langle E_x(\mathbf{r}_j)E_x^*(\mathbf{r}_j) \rangle & \langle E_x(\mathbf{r}_j)E_y^*(\mathbf{r}_j) \rangle \\ \langle E_y(\mathbf{r}_j)E_x^*(\mathbf{r}_j) \rangle & \langle E_y(\mathbf{r}_j)E_y^*(\mathbf{r}_j) \rangle \end{pmatrix}. \quad (28)$$

The main drawback is that this definition depends on the polarization state. For example, for orthogonal polarizations  $\langle \mathbf{E}(\mathbf{r}_1) \cdot \mathbf{E}^*(\mathbf{r}_2) \rangle = 0$  this degree of coherence vanishes  $\mu_1 = 0$ , even if there is perfect correlation between the fields at the apertures. More specifically, we say that  $\mu_1$  is not invariant under  $U(2) \times U(2)$  transformations, i. e., under the action of unitary  $2 \times 2$  matrices (transparent phase plates) applied to the fields at the apertures.

Two similar strategies have been proposed to solve this difficulty. On the one hand, we can consider the maximum of  $\mu_1$  when arbitrary phase plates are placed in the apertures leading to [9]

$$\mu_{1,\max} = \frac{\lambda_+ + \lambda_-}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}}, \quad (29)$$

where  $\lambda_{\pm} \geq 0$  are the singular values of  $\Gamma_{1,2}$ , i. e.

$$\Gamma_{1,2} = V_1 \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} V_2^\dagger, \quad (30)$$

where  $V_{1,2}$  are suitable unitary matrices.

On the other hand, we can consider the maximum of  $\mu_1$  when arbitrary phase plates followed by arbitrarily oriented polarizer are placed in the apertures (the same polarizer for the two apertures), leading to [18, 19]

$$\mu'_{1,\max} = \text{msv} \left( \Gamma_{1,1}^{-1} \Gamma_{1,2} \Gamma_{2,2}^{-1} \right), \quad (31)$$

where  $\text{msv}$  is the maximum singular value of the corresponding matrix .

## B. Stokes fringes in a Young interferometer

Another approach that also focus on the Young interferometer is based on the visibility of the four systems of fringes obtained by measuring the four point Stokes parameters at the observation plane, leading to [11–17]

$$\mu_2^2 = \frac{|\langle E_x(\mathbf{r}_1)E_x^*(\mathbf{r}_2)\rangle|^2 + |\langle E_x(\mathbf{r}_1)E_y^*(\mathbf{r}_2)\rangle|^2 + |\langle E_y(\mathbf{r}_1)E_x^*(\mathbf{r}_2)\rangle|^2 + |\langle E_y(\mathbf{r}_1)E_y^*(\mathbf{r}_2)\rangle|^2}{I(\mathbf{r}_1)I(\mathbf{r}_2)}, \quad (32)$$

or, equivalently,

$$\mu_2^2 = \frac{\text{tr}(\Gamma_{1,2}\Gamma_{1,2}^\dagger)}{\text{tr}\Gamma_{1,1}\text{tr}\Gamma_{2,2}} = \frac{\lambda_+^2 + \lambda_-^2}{I(\mathbf{r}_1)I(\mathbf{r}_2)}. \quad (33)$$

This definition is invariant under  $U(2) \times U(2)$  transformations.

## C. Fringes in arbitrary interferometers

Finally other approaches consider all components on an equal footing so that the degree of coherence is a function of the whole Hermitian  $4 \times 4$  correlation matrix  $\Gamma$

$$\Gamma = \begin{pmatrix} \Gamma_{1,1} & \Gamma_{1,2} \\ \Gamma_{2,1} & \Gamma_{2,2} \end{pmatrix}, \quad (34)$$

containing the sixteen matrix elements  $\langle E_j(\mathbf{r}_m)E_\ell^*(\mathbf{r}_n)\rangle$  for  $j, \ell = x, y$  and  $m, n = 1, 2$ , instead of defining it in terms of the  $2 \times 2$  complex matrix  $\Gamma_{1,2}$ . This definition suits to the idea that arbitrary interferometers mix the four field components without taking into account to which wave they belong.

In this regard we can define the degree of coherence as the distance of  $\Gamma$  to the  $4 \times 4$  identity matrix  $I_4$  representing fully incoherent and fully unpolarized light in the form [21, 22]

$$\mu_3^2 = \frac{4}{3} \text{tr} \left[ \left( \frac{1}{\text{tr}\Gamma} \Gamma - \frac{1}{4} I_4 \right)^2 \right]. \quad (35)$$

This definition is invariant under the action of  $4 \times 4$  unitary matrices, that includes the  $U(2) \times U(2)$  invariance as a particular case.

This definition is equivalent to the degree of polarization of the four-dimensional wave  $\mathbf{E} = (E_x(\mathbf{r}_1), E_y(\mathbf{r}_1), E_x(\mathbf{r}_2), E_y(\mathbf{r}_2))$ . This is interesting since in the scalar case the maximum degree of coherence that can be obtained by combining two waves  $E_{1,2}$  is the degree of polarization of the two-dimensional wave  $\mathbf{E} = (E_1, E_2)$ .

Moreover,  $\mu_3$  combines the degree of polarization of the individual waves  $P_{1,2}$  with  $\mu_2$  as

$$\mu_3^2 = \frac{I_1^2(1 + 2P_1^2) + I_2^2(1 + 2P_2^2) + 2I_1I_2(4\mu_2^2 - 1)}{3(I_1 + I_2)^2}, \quad (36)$$

where  $I_{1,2} = I(\mathbf{r}_{1,2})$  are the corresponding intensities. Finally we can mention that  $\mu_3$  provides an upper bound to the visibility  $V$  of any interferometer. For example, for two beam interferometers we have

$$\sqrt{\frac{3}{2}} \frac{I_1 + I_2}{I} \mu_3 \geq V, \quad (37)$$

where  $I$  with  $I_1 + I_2 \geq I$  is the intensity of the two interfering beams stracted from the original fields. A different visibility bound has been obtained in Ref. [34]

$$\frac{\lambda_{\max} - \lambda_{\min}}{\lambda_{\max} + \lambda_{\min}} \geq V, \quad (38)$$

where  $\lambda_{\max, \min}$  are the maximum and minimum eigenvalues of  $\Gamma$ .

A similar approach was considered in terms of the  $4 \times 4$  correlation matrix  $\mathbf{L}$  with normalized matrix elements  $\langle \epsilon_j(\mathbf{r}_m) \epsilon_\ell^*(\mathbf{r}_n) \rangle$  for  $j, \ell = x, y$  and  $m, n = 1, 2$  where

$$\epsilon_j(\mathbf{r}\cdot) = \frac{E_j(\mathbf{r}\cdot)}{\sqrt{\langle |E_j(\mathbf{r}\cdot)|^2 \rangle}}, \quad (39)$$

are normalized field components. Among othe possibilities this allows to define a degree of coherence in the form [20]

$$\mu_4^2 = \frac{1}{12} [\text{tr}(\mathbf{L}^2) - 4]. \quad (40)$$

This definition is not invariant under the action of  $4 \times 4$  unitary matrices. Moreover, due to the normalization it predicts vanishing degree of coherence for correlation matrices representing partially coherent light, i. e.,  $\mathbf{L} = \mathbf{I}_4$  does not imply  $\Gamma = \mathbf{I}_4$ .

#### D. Overall degree of coherence

Finally we can consider the Wigner ray picture of the overall degree of coherence for vectorial waves  $\mu_G$  introduced as a weighted average of the local degree of coherence  $\mu_2$  [17]

$$\mu_G^2 = \frac{\int d^2\mathbf{r}_1 d^2\mathbf{r}_2 I(\mathbf{r}_1) I(\mathbf{r}_2) \mu_2^2(\mathbf{r}_1, \mathbf{r}_2)}{[\int d^2\mathbf{r} I(\mathbf{r})]^2}, \quad (41)$$

which can be expressed in terms of the Wigner function as [4, 33]

$$\mu_G^2 = \frac{8\pi^3 \int d^2\mathbf{r} d^2\mathbf{p} d^2O W^2(\mathbf{r}, \mathbf{p}, O)}{k^2 [\int d^2\mathbf{r} d^2\mathbf{p} d^2O W(\mathbf{r}, \mathbf{p}, O)]^2}, \quad (42)$$

or, equivalently,

$$\mu_G^2 = \frac{8\pi^4}{k^2} \frac{\int d^2\mathbf{r} d^2\mathbf{p} S^2(\mathbf{r}, \mathbf{p})}{[\int d^2\mathbf{r} d^2\mathbf{p} S_0(\mathbf{r}, \mathbf{p})]^2}, \quad (43)$$

where  $d^2O = \sin\theta d\theta d\phi$ .

## V. VECTORIAL COHERENCE AND WIGNER FUNCTION

As in the scalar case we can express the degree of coherence in terms of the Wigner function for vectorial waves from two different perspectives.

### A. Coherence as phase-difference average

On the one hand, the degrees of coherence  $\mu_1$  and  $\mu_2$  at two points  $\mathbf{r}_{1,2}$  are averages of the phase difference  $\varphi = k\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)$  [33]

$$\mu_1 = \frac{|\langle \exp(i\varphi) \rangle_{S_0}|}{\sqrt{I_1 I_2}}, \quad \mu_2 = \frac{|\langle \exp(i\varphi) \rangle_S|}{\sqrt{I_1 I_2}}, \quad (44)$$

where the weights are now ray-Stokes parameters at the midpoint  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ . More specifically for  $\mu_1$  the weights are the first Stokes parameters  $S_0$  and for  $\mu_2$  the weights are the four ray-Stokes parameters  $\mathbf{S}$

$$\begin{aligned} \langle \exp(i\varphi) \rangle_{S_0} &= \int d^2\mathbf{p} e^{ik\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} S_0(\mathbf{R}, \mathbf{p}), \\ \langle \exp(i\varphi) \rangle_S &= \int d^2\mathbf{p} e^{ik\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)} \mathbf{S}(\mathbf{R}, \mathbf{p}), \end{aligned} \quad (45)$$

so that  $\langle \exp(i\varphi) \rangle_S$  vectorial quantity.

#### 1. Coherence in a Young interferometer

On the other hand, the ray picture of vectorial coherence in action in the Young interferometer reproduces the scalar case replacing Wigner function by Stokes parameters. After the apertures there are three secondary sources of rays. Two real sources at the apertures that emit isotropically bright rays with ray Stokes parameters proportional to the point Stokes parameters at the apertures  $\mathbf{S}(\mathbf{r}_{1,2}, \mathbf{p}) = \mathbf{S}(\mathbf{r}_{1,2}) \propto \mathbf{s}(\mathbf{r}_{1,2})$  and independent of  $\mathbf{p}$ , and a fictitious source of bright and dark rays at the midpoint between the apertures  $\mathbf{S}(\mathbf{R}, \mathbf{p})$  that provides the interference term. Each observation point  $\mathbf{r}$  in a plane at a distance  $z$  from

the plane of the apertures is reached exclusively by three rays, one from each source. Their incoherent superposition gives the point Stokes parameters at the observation plane [22, 33]

$$\mathbf{s}(\mathbf{r}) \propto \mathbf{S}(\mathbf{r}_1) + \mathbf{s}(\mathbf{r}_2) + \mathbf{S}(\mathbf{R}, \mathbf{p}), \quad (46)$$

with  $\mathbf{p} = n \frac{\mathbf{r} - \mathbf{R}}{z}$ .

The degrees of coherence can be then expressed in terms of the ray Stokes parameters at the midpoint source following relations analogous to the case in the form [33]

$$\mu_1^2 = \frac{S_{0,\max}^2(\mathbf{R}, \mathbf{p})}{4S_0(\mathbf{r}_1)S_0(\mathbf{r}_2)}, \quad \mu_2^2 = \frac{\mathbf{S}_{\max}^2(\mathbf{R}, \mathbf{p})}{8S_0(\mathbf{r}_1)S_0(\mathbf{r}_2)}, \quad (47)$$

and also

$$\mu_3^2 = \frac{2\mathbf{S}^2(\mathbf{r}_1) + 2\mathbf{S}^2(\mathbf{r}_2) + \mathbf{S}_{\max}^2(\mathbf{R}, \mathbf{p})}{3[S_0(\mathbf{r}_1) + S_0(\mathbf{r}_2)]^2} - \frac{1}{3}, \quad (48)$$

where  $S_{j,\max}(\mathbf{R}, \mathbf{p})$  refers to the maximum value when  $\mathbf{p}$  is varied. The maximum can occur for a different  $\mathbf{p}$  for each component  $j$ .

Moreover, the degree of coherence  $\mu_2$  can be expressed also as the amount of Wigner function located at the midpoint

$$\mu_2^2 = \frac{k^2}{\pi^3} \frac{\int d^2\mathbf{p} d^2\mathbf{r} d^2O W^2(\mathbf{p}, \mathbf{r}, O)}{S_0(\mathbf{r}_1)S_0(\mathbf{r}_2)}. \quad (49)$$

Furthermore, this allows us to relate the degrees of coherence with measurable quantities as in [15, 16]. This is because the measurement of the point Stokes parameters at the apertures  $\mathbf{S}(\mathbf{r}_{1,2})$  and at the interference plane  $\mathbf{s}(\mathbf{r})$  determine the interference term  $\mathbf{S}(\mathbf{R}, \mathbf{p})$  and the different degrees of coherence via the above relations.

## B. Example

We illustrate this approach with a simple example of a Young interferometer illuminated by unpolarized partially coherent light [10, 22]

$$\Gamma_{1,2} = \begin{pmatrix} B_x & 0 \\ 0 & B_y \end{pmatrix}, \quad \Gamma_{1,1} = \Gamma_{2,2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (50)$$

where for simplicity  $B_{x,y}$  are assumed real and positive.

The point Stokes vector  $(s_1, s_2, s_3)$  and the degree of polarization  $P$  vanish throughout the plane of the aperture. On the other hand, the ray Stokes vector  $(S_1, S_2, S_3)$  does not

vanish on the midpoint, where for example we have  $\mathbf{S}(\mathbf{R}, \mathbf{p} = 0) \propto (B_x + B_y, B_x - B_y, 0, 0)$ . This nonvanishing contribution generates point polarization in the interference plane since  $s_1 \propto S_1$ , leading to a degree of polarization in the vertical of the midpoint

$$P = \frac{|B_x - B_y|}{2 + B_x + B_y}. \quad (51)$$

This degree of polarization is due exclusively to the contribution of the ray from the midpoint source.

Concerning the degrees of coherence we have for this example

$$\mu_1 = \mu_{1,\max} = \frac{B_x + B_y}{2}, \quad \mu'_{1,\max} = \max(B_x, B_y), \quad (52)$$

while  $\mu_2$  and  $\mu_3$  are proportional,

$$\mu_3 = \sqrt{\frac{2}{3}}\mu_2 = \sqrt{\frac{B_x^2 + B_y^2}{6}}, \quad (53)$$

because of the lack of polarization of the illuminating beam.

## VI. CONCLUSIONS

The main interest of the approach elaborated in this work is the unification of wave and geometrical ideas in a single formalism. Coherence phenomena can be embodied in a geometrical picture if we allow the rays to transport Wigner function or Stokes parameters. This is equivalent to a complete geometrical formulation of paraxial optics that includes coherence as a phase-difference average or as negativity of the Wigner function.

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