

Finite-mode analysis by means of intensity information in fractional optical systems

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Received April 12, 2001; revised manuscript received July 11, 2001; accepted July 26, 2001

It is shown how a coherent optical signal that contains only a finite number of Hermite–Gauss modes can be reconstructed from the knowledge of its Radon–Wigner transform—associated with the intensity distribution in a fractional-Fourier-transform optical system—at only two transversal points. The proposed method can be generalized to any fractional system whose generator transform has a complete orthogonal set of eigenfunctions. © 2002 Optical Society of America

OCIS codes: 100.5070, 070.1170, 070.2590, 070.6020, 200.1130.

1. INTRODUCTION

Phase retrieval from intensity distributions is a very old problem, which can be solved in different ways (see for example, Refs. 1–5). In order to create a more optimal algorithm, all possible additional knowledge about an optical signal has to be taken into account. In this paper we propose a new method of phase reconstruction, which can be applied in the case of a coherent field containing a finite number of Hermite–Gauss modes.

It was shown^{6,7} that the Hermite–Gauss-mode content of an optical signal can be determined from the knowledge of the complex field amplitude propagating through a fractional Fourier transform (FT) system when measured at one transversal point. In this paper we show that, under the assumption that the number of modes is finite, the mode content can be determined from the evolution of the intensity distribution in the fractional FT system^{8,9} taken at two transversal points. This implies that the phase of a coherent field with $N + 1$ modes can be reconstructed from measurements of the intensity distributions at $2(N + 1)$ sensor points taken over two lines parallel to the optical axis of the fractional FT system.

Recently several methods of phase retrieval based on the properties of the fractional FT were proposed: Iterative and recursive algorithms for phase recovery from the intensity distributions in two fractional FT domains were discussed.^{4,5} In the present paper we consider the possibility of phase retrieval from the evolution of the intensity distribution in the fractional FT system along two lines parallel to the optical axis. Therefore our method of phase retrieval is based on initial information other than the Gerchberg–Saxton algorithm⁴ or the recursive algorithm for phase retrieval in the fractional FT domain,⁵ both of which use two fractional Fourier power spectra for phase reconstruction. Moreover, this approach can be generalized to any fractional transform system¹⁰ whose generator transform has a complete orthogonal set of eigenfunctions (modes). As a useful example of such a transform in optics, we mention the fractional Hankel transform with Laguerre–Gauss eigenfunctions.

2. INTENSITY EVOLUTION UNDER FRACTIONAL FOURIER TRANSFORMATION

Fractional FT optical systems^{8,9} have been actively discussed over the past decade as useful tools for optical signal analysis. The evolution of the complex field amplitude in the paraxial approximation of the scalar diffraction theory during propagation through a quadratic refractive-index medium is described by the fractional FT.

The fractional FT of a function $f(x)$ can be defined as¹¹

$$R^\alpha[f(x)](u) = F_\alpha(u) = \int_{-\infty}^{\infty} K(\alpha, x, u)f(x)dx, \quad (1)$$

where the kernel $K(\alpha, x, u)$ is given by

$$K(\alpha, x, u) = \frac{\exp[i(\alpha/2)]}{\sqrt{i2\pi \sin \alpha}} \exp\left[i \frac{(x^2 + u^2)\cos \alpha - 2ux}{2 \sin \alpha}\right]. \quad (2)$$

Note that, in particular, $F_0(u) = f(u)$ and that $F_{\pi/2}(u)$ corresponds to the normal FT of $f(x)$. The fractional FT is continuous, additive, and periodic with respect to the parameter α , which can be regarded as a rotation angle in the phase plane. Note, moreover, that we use dimensionless variables x and u .

The fractional power spectrum, i.e., the squared modulus of the fractional FT $|F_\alpha(u)|^2$ associated with an intensity distribution, is also periodic in α and can thus be represented as a Fourier series,

$$|F_\alpha(u)|^2 = \sum_{n=-\infty}^{\infty} Q_n(u)\exp(in\alpha), \quad (3)$$

where

$$Q_n(u) = \frac{1}{2\pi} \int_0^{2\pi} |F_\alpha(u)|^2 \exp(-in\alpha) d\alpha. \quad (4)$$

Since $|F_\alpha(u)|^2$ is a real function, its FTs with respect to the variables u or α are Hermitian; in particular, we thus

have $Q_{-n}(u) = Q_n^*(u)$, where the asterisk * denotes complex conjugation. Note that the knowledge of the two-dimensional function $|F_\alpha(u)|^2$ of α and u , which is called the Radon–Wigner transform, permits us to reconstruct the Wigner distribution of the signal and then to determine its phase² or, in the case of a partially coherent field, its two-point correlation function. In this paper we introduce a method that significantly reduces the number of measurements, under the assumption that the signal is coherent and contains a finite number of modes.

The Hermite–Gauss functions $\Psi_n(x)$, defined by

$$\Psi_n(x) = \frac{1}{(2^n n! \sqrt{\pi})^{1/2}} \exp\left(\frac{-x^2}{2}\right) H_n(x), \quad (5)$$

with $H_n(x)$ the Hermite polynomials, form a complete orthonormal set $\{\Psi_n(x), n = 0, 1, \dots, \infty\}$ on the entire x axis. We can thus represent a function $f(x)$ in the form

$$f(x) = \sum_{n=0}^{\infty} f_n \Psi_n(x). \quad (6)$$

Since the Hermite–Gauss functions are the eigenfunctions for the fractional FT operator with eigenvalue $\exp(-in\alpha)$,

$$K(\alpha, x, u) = \sum_{n=0}^{\infty} \Psi_n^*(x) \Psi_n(u) \exp(-in\alpha), \quad (7)$$

the fractional FT of $f(x)$ can be expressed as [cf. Eq. (6)]

$$F_\alpha(u) = \sum_{n=0}^{\infty} f_n \exp(-in\alpha) \Psi_n(u). \quad (8)$$

3. FINITE NUMBER OF MODES

Suppose that an optical signal contains only a finite number of $N + 1$ successive modes. This implies that there are not more than $N + 1$ nonzero coefficients f_n in mode expansion (6) and that we can write

$$f(x) = \sum_{n=L}^{L+N} f_n \Psi_n(x) \quad (9)$$

with L ($L \geq 0$) the number of the lowest-order, nonzero mode coefficient f_L and with $L + N$ ($N \geq 0$) the number of the highest-order, nonzero mode coefficient f_{L+N} . The fractional FT now takes the form [cf. Eq. (8)]

$$F_\alpha(u) = \sum_{n=L}^{L+N} f_n \exp(-in\alpha) \Psi_n(u), \quad (10)$$

and when we substitute this expression into Eq. (4), it is easy to see that

$$Q_n(u) = Q_{-n}^*(u) = \sum_{m=L}^{L+N-n} f_m f_{m+n}^* \Psi_m(u) \Psi_{m+n}^*(u) \quad (n = 0, \dots, N), \quad (11)$$

where $Q_n(u) = 0$ for $|n| > N$. Note that in general $Q_n(u)$ has a complex value, except for $n = 0$, for which its value is real.

If the coefficients $Q_n(u)$ ($n = 0, \dots, N$) are considered for only one value of u , the system (11) contains $2N + 1$ real equations for $2N + 2$ real variables $\text{Re}f_n$ and $\text{Im}f_n$ ($n = L, \dots, L + N$). Nevertheless, to solve this system is difficult owing to its nonlinearity. If the coefficients $Q_n(u)$ ($n = 0, \dots, N$) are known at two points u_1 and u_2 such that $\Psi_n(u_1) \neq 0$ and $\Psi_n(u_2) \neq 0$ for $n = L$ and for $n = L + N$, the system (11) can be transformed into a recursive set of N subsystems consisting of two linear equations for two variables, which permits us to determine the mode coefficients f_{L+n} ($n = 0, \dots, N$) up to a constant phase factor.

To show this, we first define the normalized mode coefficients

$$\xi_m = \frac{f_{L+m}}{f_L}, \quad \eta_m = \frac{f_{L+N-m}}{f_{L+N}} \quad (m = 1, \dots, N) \quad (12)$$

and the normalized constants

$$q_m(u) = \frac{Q_{N-m}(u)}{Q_N(u)}, \quad \psi_m(u) = \frac{\Psi_{L+m}(u)}{\Psi_L(u)},$$

$$\phi_m(u) = \frac{\Psi_{L+N-m}^*(u)}{\Psi_{L+N}^*(u)} \quad (m = 1, \dots, N). \quad (13)$$

Note that the mode coefficients $f_{L+m} = \xi_m f_L$ ($m = 1, \dots, N$) follow from the normalized mode coefficients ξ_m ($m = 1, \dots, N$) and the lowest-order, nonzero mode coefficient f_L .

To determine the normalized mode coefficients ξ_m and η_m ($m = 1, \dots, N$), we divide all rows (except the $n = N$ one) of the system (11) by the $n = N$ row to get

$$\xi_k \psi_k(u) + \eta_k^* \phi_k(u) = q_k(u) - c_k(u), \quad (k = 1, \dots, N) \quad (14)$$

with

$$c_1(u) = 0,$$

$$c_k(u) = \sum_{m=1}^{k-1} \xi_m \eta_{k-m}^* \psi_m(u) \phi_{k-m}(u) \quad (k = 2, \dots, N). \quad (15)$$

It is easy to see that every next equation of the system (14), let us say the k th one, contains only two new mode coefficients ξ_k and η_k compared with the $k - 1$ previous equations, and the system can thus be solved recursively. Note that from their definitions (12) it follows that $\xi_N = \xi_m / \eta_{N-m} = 1 / \eta_N$; the highest-order normalized mode coefficient $\xi_N = 1 / \eta_N$ thus follows directly from any two ξ_m and η_{N-m} ($m = 1, \dots, N - 1$), and the last equation of the system (14), the one for $k = N$, could in fact be omitted.

If we consider the system (14) for two points u_1 and u_2 , all the normalized mode coefficients ξ_k ($k = 1, \dots, N$) can be consecutively recovered from the system of linear equations

$$\begin{bmatrix} \psi_k(u_1) & \phi_k(u_1) \\ \psi_k(u_2) & \phi_k(u_2) \end{bmatrix} \begin{pmatrix} \xi_k \\ \eta_k^* \end{pmatrix} = \begin{pmatrix} q_k(u_1) - c_k(u_1) \\ q_k(u_2) - c_k(u_2) \end{pmatrix} \quad (k = 1, \dots, N), \quad (16)$$

provided that u_1 and u_2 are chosen such that each 2×2 -matrix constructed from $\psi_k(u_1)$, $\phi_k(u_1)$, $\psi_k(u_2)$, and $\phi_k(u_2)$ ($k = 1, \dots, N$) is nonsingular.

Furthermore, from the $n = N$ row of the system (11) and substituting from the definitions (12), we have

$$\frac{Q_N(u)}{\Psi_L(u)\Psi_{L+N}^*(u)} = |f_L|^2 \xi_N^*, \quad (17)$$

and we can determine the lowest-order, nonzero mode coefficient f_L up to a constant phase factor. We can thus conclude that if the intensity distributions $|F_\alpha(u_1)|^2$ and $|F_\alpha(u_2)|^2$ along two lines parallel to the optical axis are measured, all $N + 1$ complex mode coefficients f_n ($n = L, \dots, L + N$), and hence the signal itself, can be determined up to this common constant phase factor.

Note that the constants $q_k(u)$, $\psi_k(u)$, and $\phi_k(u)$ depend on the number L of the lowest-order, nonzero mode. If this number is known in advance, the procedure described above can be applied without any problem. In the case that this number is not known in advance, the procedure can be applied with the arbitrary choice $L = L_0$ first. If the reconstructed intensity distribution does not fit to the measured intensity distribution $|f(x)|^2$ for $L = L_0$, we have to repeat the procedure for other values of L until a proper fit will occur.

In summary, the proposed algorithm can be divided into the following steps:

1. Calculate $Q_n(u_1)$ and $Q_n(u_2)$ ($n = 0, \dots, N$) as the Fourier transforms with respect to α of the intensity measurements $|F_\alpha(u_1)|^2$ and $|F_\alpha(u_2)|^2$ [see Eq. (4)]. We assume that $|F_\alpha(u)|^2$ is a band-limited signal and that its Fourier spectrum does not contain more than $2N + 1$ nonzero elements: $Q_n(u) = 0$ for $|n| > N$. Note that N can always be found if the number of sensor points is sufficiently high; then the cutoff frequency of the Fourier transform of $|F_\alpha(u)|^2$ forms an indication of N .

2. With L known—or arbitrarily set to its lowest possible value $L = L_0$ —form the normalized constants $q_m(u_i)$, $\psi_m(u_i)$, and $\phi_m(u_i)$ ($i = 1, 2$; $m = 1, \dots, N$) according to Eq. (13).

3. Determine the normalized coefficients ξ_k and η_k^* ($k = 1, \dots, N$) by recursively solving Eq. (16), starting from $k = 1$ and $c_1(u_1) = c_1(u_2) = 0$.

4. Find the mode coefficient f_L up to a constant phase factor using Eq. (17).

5. Determine the mode coefficients f_{L+m} using Eq. (12) for $m = 1, \dots, N$.

Note that for the implementation of the algorithm the range of the modes (with lower bound L and upper bound $L + N$) has to be known *a priori*. In the case that only the number of modes N is known, but not the lower bound L , an initial value $L = L_0$ should be chosen in step 2 to start the algorithm, and the algorithm has to be completed by some additional steps and some additional measurements of the intensity distributions, for example, $|f(x)|^2$. The additional calculations are then the following:

6. Calculate $|f(x_i)|^2$ for several points x_i and compare them with the measured values. In the case that they do not fit, replace L by $L + 1$ and repeat steps 2–6 until a proper fitting occurs.

In order to calculate the coefficients, we have to choose, in an optimal way, the positions of the sensors along the two lines parallel to the optical axis and the values u_1 and u_2 transversal to the optical axis. To perform the simplest algorithm for the calculation of the FT (step 1), the sensor positions α_k ($k = 0, \dots, M - 1$) along the two lines parallel to the optical axis have to be equally spaced over α , the angle parameter of the fractional FT. This implies that $\alpha_k = 2\pi k/M$, where M is the number of sensors. Although, theoretically, the algorithm could be implemented for $M = N + 1$ sensor points over each line, the occurrence of noise requires a larger number M ; usually M has to be more than $2(N + 1)$. The choice of the transversal points u_1 and u_2 is based on the following considerations. First, they have to be close to the optical axis, where the paraxial approximation of the diffraction theory applies. Moreover, in order to solve the system (16), the Hermite–Gauss functions $\Psi_n(u_i)$ ($i = 1, 2$; $n = L, \dots, L + N$) should never lead to a singular 2×2 matrix. Finally, use of the normalized constants (13) requires that $\Psi_n(u_1)$ and $\Psi_n(u_2)$ for $n = L$ and $n = L + N$ differ from zero.

The computational complexity of the main algorithm (steps 1–5) is composed of (i) two times a calculation of the FT (corresponding to $2M \log_2 M$ multiplication operations if the fast FT is used, where M is the number of sensor points along the lines parallel to the optical axis), (ii) the calculation of $N + 1$ Hermite–Gauss functions at two points (leading to $N^2 + 5N$ multiplication operations), and (iii) the normalization procedure, the solution of the system of linear equations, and the determination of the mode coefficients (corresponding to $12N$ multiplication operations).

As an example let us consider the reconstruction of the optical field described by

$$f(x) = 0.54x^4 \exp(-x^2/2)(1 + ix^2\sqrt{3}), \quad (18)$$

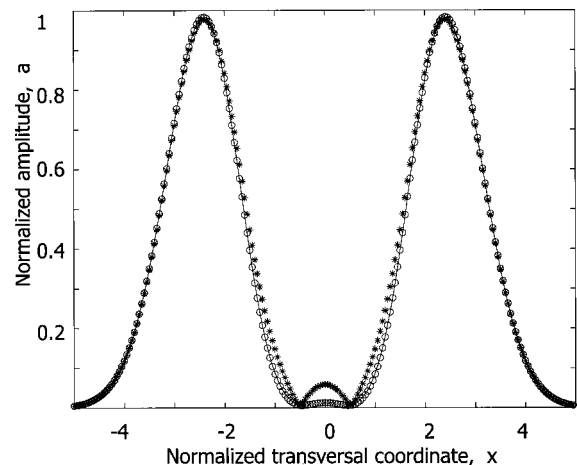
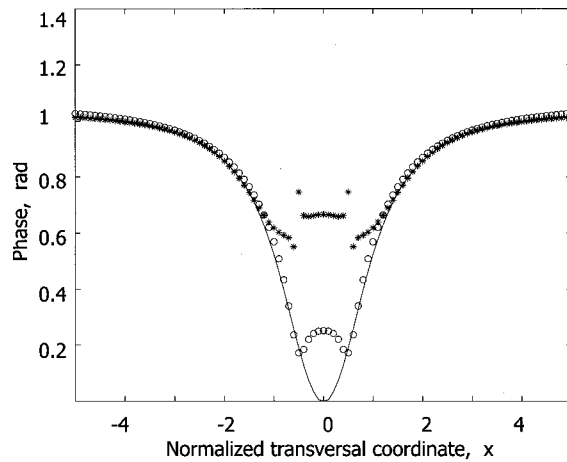


Fig. 1. Amplitude of $f(x)$.

Fig. 2. Phase of $f(x)$.

for which the lowest-order, nonzero Hermite–Gauss mode is $\Psi_0(x)$ and the highest-order is $\Psi_6(x)$; hence $L = 0$ and $N = 6$. The evolution of the intensity distribution taken in the transversal points $u_1 = -1$ and $u_2 = 0$ was chosen for further processing; the number of sensor points along the lines parallel to the optical axis was taken equal to 16. The intensity distribution was corrupted by additive white Gaussian noise with an intensity of 5% of its current value. The amplitude and the phase of the original and reconstructed signals are represented in Figs. 1 and 2, respectively. The solid curve in these figures corresponds to the original function, while the signs * and o correspond respectively to the function reconstructed from one particular realization and from the average of 30 realizations. Note that the phase might not be reconstructed accurately if the amplitude is low.

4. CONCLUSIONS

Although this paper was restricted to the reconstruction of a signal with a finite number of Hermite–Gauss functions, the same method can be used in the more general case when a signal can be decomposed into a finite set of orthogonal functions that are the eigenfunctions of a cyclic transform, i.e., a transform that produces the identity transform after M consecutive operations. It was shown¹⁰ that one of the possibilities for defining a fractional transform that is associated with a given cyclic generator transform has as its kernel

$$K(\alpha, x, u) = \sum_{n=0}^{\infty} \Phi_n^*(x) \Phi_n(u) \exp(in\alpha l), \quad (19)$$

where $\{\Phi_n(x)\}$ is the orthonormal set of the eigenfunctions of the generator transform $K(2\pi/M, x, u)$ and l is an integer. For the fractional FT defined by Eq. (2), $l = -1$ and $\Phi_n(x)$ are the Hermite–Gauss functions; for the fractional Hankel transform, $l = +1$ and $\Phi_n(x)$ are the Laguerre–Gauss functions.¹² If a signal now contains only a finite number of the modes $\Phi_n(x)$, it can be reconstructed from the evolution of the intensity distributions in the corresponding fractional system taken at two transversal points. A useful example of the fractional optical system is the fractional Hankel-transform system: It permits the reconstruction of the Laguerre–Gauss mode content of rotationally symmetric optical beams.

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