Two properties of the electromagnetic knots

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Abstract

We prove two properties of the electromagnetic knots in empty space. First, that any standard radiation fields (*i. e.* verifying $\mathbf{E}\cdot\mathbf{B}=0$) coincide locally with an electromagnetic knot. Second, that the electric and magnetic helicities of any knot are equal. These results can be used as a basis for a topological model of electromagnetism.

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1 Introduction

Electromagnetic knots are very curious solutions of the standard classical Maxwell equations. Their defining property is that any pair of magnetic lines or any pair of electric lines is a link [1, 2, 3, 4, 5], characterized by the integer n_m or n_e, respectively, which gives the value of the corresponding helicity [6, 7, 8], and can be interpreted as Hopf invariants [9, 10]. They are interesting not only on their own; recently, they have been used as a basis for a model of ball lightning [4]. In this last case, it was shown that a linked magnetic bottle —*i. e.* a magnetic knot coupled to a plasma— is more efficient to confine a plasma than an unlinked one because the confinement time is bigger if there is linking. We may mention that Kamchatnov had considered what he called a topological soliton which corresponds to our $n_m = 1$, $n_e = 0$, coupled to a plasma, and concluded that it expands because of the Joule effect [11].

In this work we study further the idea of electromagnetic knot, proving the following two properties in the case of empty space: (i) any standard classical electromagnetic field of radiation type (also called singular or degenerate) is locally equal to an electromagnetic knot; (ii) the electric and magnetic helicities of any knot are equal. The plan of the paper is as follows.

In section 2 we prove property (i); in section 3, property (ii), and give the explicit time dependent expressions of a family of knots. Later, in section 4, we show that it is possible to formulate a topological model which is locally equivalent to classical Maxwell theory and is based on a variational principle. Finally, in section 5 we state the conclusions.

2 Radiation electromagnetic fields are locally equal to electromagnetic knots

In reference [3] a method to construct electromagnetic knots was proposed. A slightly more formal procedure is the following.

Let $\phi(\mathbf{r}, t)$, $\theta(\mathbf{r}, t)$ be a couple of regular classical complex scalar fields; then, via stereographic projection, the one-point compactification of the complex numbers can be identified with the sphere S^2, so that they give regular maps $\phi, \theta : M = R^3 \times R \to S^2$, M being the Minkowski space-time. Let σ be the normalized area 2-form on S^2. Suppose now that ϕ and θ are dual in the sense of the equality

$$-*(\phi^*\sigma) = \theta^*\sigma \ , \tag{1}$$

where $\theta^*\sigma$, $\phi^*\sigma$ are the corresponding pull-backs of σ and $*$ is the Hodge or duality operator in the Minkowski space-time, which verifies $*^2 = -1$ (see sections 2 and 3 of [2].) If we define the 2-forms $\mathcal{F}$ and $*\mathcal{F}$ as

$$\mathcal{F} = -\sqrt{a}\,\phi^*\sigma \ , \ *\mathcal{F} = \sqrt{a}\,\theta^*\sigma \ , \tag{2}$$

where a is a normalizing constant with dimensions of action times velocity (we use $\sqrt{a}$ instead of a for convenience), it turns out that $\mathcal{F}$ and $*\mathcal{F}$ obey necessarily Maxwell equations in empty space:

$$d\mathcal{F} = 0 \ , \ d*\mathcal{F} = 0 \ , \tag{3}$$

because $d\phi^*\sigma = \phi^* d\sigma = 0$ since σ is the area form in S^2. They define an electromagnetic wave, having the right dimensions to be the Faraday form and its dual. In references [1, 2], some examples of couples (ϕ, θ) are given.

We require ϕ, θ to be single-valued at infinity, so that the electromagnetic wave has finite energy. This makes possible compactifying R^3 to S^3 and interpreting the scalars as giving two maps $S^3 \to S^2$ at every time. The consequence is important, since the magnetic and electric helicities are then topological constants of the motion, that is

$$h_m = \int \mathbf{A}\cdot\mathbf{B}\,d^3r = H(\phi)a \ , \ h_e = \int \mathbf{C}\cdot\mathbf{E}\,d^3r = H(\theta)a \ , \tag{4}$$

where $\mathbf{A}$ and $\mathbf{C}$ are vector potentials for $\mathbf{B}$ and $\mathbf{E}$, $H(\phi)$ and $H(\theta)$ are two integer numbers called the Hopf invariants of the maps $\phi, \theta : S^3 \to S^2$ and the integrals extend to all R^3. The corresponding waves have been called *electromagnetic knots* [1, 3, 5] because any pair of magnetic lines and any pair of electric lines is a link with linking numbers $H(\phi)$ and $H(\theta)$, respectively.

Not all the solutions of classical Maxwell equations in vacuum can be written as (2) in terms of a pair of scalars. However, there is a local equivalence of these electromagnetic knots with the standard theory as we will see now.

It is known that, because of the Darboux theorem [12], the Faraday form of any electromagnetic field $\mathcal{F}$, and its dual $*\mathcal{F}$, can be written, locally, as

$$\mathcal{F} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2, \quad *\mathcal{F} = dv_1 \wedge du_1 + dv_2 \wedge du_2, \tag{5}$$

where q_k, p_k, v_k, u_k are functions of spacetime and also good canonical variables and coordinates of the field [13]. Each one of the two terms in these sums is a radiation field (*i. e.* verifies $\mathbf{E} \cdot \mathbf{B} = 0$). This means that any standard electromagnetic field in empty space can be expressed as the sum of two fields of radiation type, although it must be said that this representation is not unique (by standard electromagnetic field we mean any solution of Maxwell equations). Consequently, a pure radiation field is expressable as

$$\mathcal{F} = dq \wedge dp, \quad *\mathcal{F} = dv \wedge du, \tag{6}$$

the electric and magnetic fields having then the form

$$\mathbf{B} = \nabla p \times \nabla q, \ \ \mathbf{E} = \nabla u \times \nabla v. \tag{7}$$

The functions (p, q) and (u, v) are called Clebsh variables of $\mathbf{B}$ and $\mathbf{E}$, respectively. They are not uniquely defined, but can be changed by canonical transformations.

However, it will be more convenient for our purpose to redefine these variables, writing instead

$$\mathcal{F} = \sqrt{a}\, dq \wedge dp, \quad *\mathcal{F} = \sqrt{a}\, dv \wedge du. \tag{8}$$

In this case, $\mathbf{B}$, $\mathbf{E}$ can be expressed as

$$\begin{aligned} \mathbf{B} &= \sqrt{a}\, \nabla p \times \nabla q = \sqrt{a}\, (\partial_0 u \nabla v - \partial_0 v \nabla u) \\ \mathbf{E} &= \sqrt{a}\, \nabla u \times \nabla v = \sqrt{a}\, (\partial_0 q \nabla p - \partial_0 p \nabla q) \end{aligned} \tag{9}$$

Note that, given a constant a, any electromagnetic field can be written in this form; in this way, the Clebsh variables are dimensionless quantities.

A very important property is satisfied by the electromagnetic knots. In a precise way, the following proposition holds true.

Any standard radiation electromagnetic field in empty space with Faraday 2-form $\mathcal{F}^{st}$, regular in a bounded spacetime domain D, coincides locally with a knot around any point $P \in D$ in the following sense: There is a knot with 2-form $\mathcal{F}^{kn}$, such that $\mathcal{F}^{st} = \mathcal{F}^{kn}$ around P, except perhaps in a zero measure set. The same property holds for $*\mathcal{F}^{st}$.

This means that the difference between the set of the radiation solutions of the Maxwell equations and the set of the electromagnetic knots is not local but global. In other words: Radiation fields and knots are locally equal. A proof is the following.

Let the Faraday 2-form of the standard radiation field $\mathcal{F}^{st}$ be expressed as (8) with p, q dimensionless. Define then the functions η, δ as

$$\eta = \pi(p^2 + q^2), \quad \delta = \frac{1}{2\pi} \arctan \frac{q}{p}. \tag{10}$$

It is then clear that

$$\mathcal{F}^{st} = \sqrt{a}\, d\delta \wedge d\eta, \tag{11}$$

so that η and δ give another election of the Clebsh variables of the standard field (they are obtained from p, q through a canonical transformation.) If an electromagnetic knot derives from the scalar $\phi = S \exp(i2\pi\gamma)$ through eqn. (2), it is easy showing that

$$\mathcal{F}^{kn} = \sqrt{a}\, d\gamma \wedge d \frac{1}{(1 + S^2)}. \tag{12}$$

This means that $\mathcal{F}^{st}$ will be a knot if there exist regular functions $S(\mathbf{r})$, $\gamma(\mathbf{r})$, one-valued at infinity, such that

$$\eta = \frac{1}{1 + S^2}, \quad \delta = \gamma. \tag{13}$$

The second equation poses no problem because δ was defined as a phase function. If $\eta < 1$, the solution for S is trivial, the standard field with form $\mathcal{F}^{st}$ being then a knot. The same happens if η is bounded, say if $\eta < A$, because we can then take as the Clebsh variables $\eta' = \eta/n'$, $\delta' = n'\delta$, n' being an integer greater than A. Dropping the primes and entering the new Clebsh variables in (13), it is clear that there exists then a solution for S, γ.

Let us take now the case in which η is not bounded in D (but $F^{st}_{\mu\nu}$ is continuous and the equation (11) is still valid). Let Σ be the three–dimensional set in which η diverges (a zero measure set). In general $D - \Sigma$ consists in k connected open components D_j. Let $D^*_j \subset D_j$ be k open subsets in which η is bounded. In each one of them, we define Clebsh variables η', δ', by the same method as before. It follows that the field is equal to a knot in each D^*_j. Now, the volume of $D - \bigcup D^*_j$ may be made as small as desired. This means that the magnetic field can be obtained by patching together those of the knots $\mathcal{F}^{kn}_j$, each one defined in the corresponding D^*_j, except for a set as small as required containing Σ. (Note that there is no problem if any D_j is not simply connected.) The same can be said of the dual to the Faraday 2-form $*\mathcal{F}^{st}$, which coincides with the corresponding 2-form of a knot, except perhaps in a zero measure set Σ'. This means that any radiation electromagnetic field coincides locally with an electromagnetic knot, except perhaps on a zero measure set. In other words, standard radiation fields can be obtained by patching together electromagnetic knots generated by ϕ_j, θ_j, each one defined in a different domain, except at most on the zero measure set $\Sigma \bigcup \Sigma'$. This ends the proof.

It is convenient now to give examples of the expression of electromagnetic fields in the form (2) or equivalently (8)-(9). We present now three: the Coulomb potential, a plane wave and a standing wave.

If $\phi = Pe^{i2\pi q}$, $\theta = Ve^{i2\pi u}$, it is easy to see that

$$p = \frac{1}{1+P^2}, \quad v = \frac{1}{1+V^2},$$

q and u being the other two Clebsh variables.

(i) *Coulomb potential*, $\mathbf{E} = Q\mathbf{r}/(4\pi r^3)$, $\mathbf{B} = 0$.

This field can be obtained from the scalars

$$\phi = \frac{ct}{r} \exp\left(iQ \frac{(c^2t^2+r^2)^2}{4c\sqrt{a}r^3t} \log(r/r_0) \right), \quad \theta = \tan\frac{\beta}{2} \exp\left(iQ \frac{\alpha}{\sqrt{a}} \right), \tag{14}$$

where α, β are the azimuth and the polar angle and r_0 is any length. The Clebsh variables are

$$p = \frac{r^2}{(r^2+c^2t^2)}, \quad q = Q\frac{(r^2+c^2t^2)^2}{8\pi c\sqrt{a}r^3t}\log(r/r_0), \quad v = \cos^2\frac{\beta}{2}, \quad u = \frac{Q\alpha}{2\pi\sqrt{a}}.$$

As can be seen, both scalars are regular everywhere except at $r = 0$ and $r = \infty$.

(ii) *Plane wave*, $\mathbf{E} = E_0(0, \sin\omega(x/c-t), 0)$, $\mathbf{B} = E_0(0, 0, \sin\omega(x/c-t))$.

The two scalars and the corresponding Clebsh variables are

$$\begin{aligned} \phi &= \frac{1+\cos\omega(x/c-t)}{\sin\omega(x/c-t)} \exp\left(i\frac{4\pi c E_0 y}{\sqrt{a}\omega} \right), \\ \theta &= \frac{1+\cos\omega(x/c-t)}{\sin\omega(x/c-t)} \exp\left(i\frac{4\pi c E_0 z}{\sqrt{a}\omega} \right), \end{aligned} \tag{15}$$

$$p=\frac{1}{2}(1-\cos\omega(x/c-t)),\ q=\frac{2cE_0y}{\sqrt{a}\omega},\ v=\frac{1}{2}(1-\cos\omega(x/c-t)),\ u=\frac{2cE_0z}{\sqrt{a}\omega}.$$

It is seen that ϕ and θ do not represent smooth maps $S^3\mapsto S^2$ because they are not well defined at infinity. However there are smooth maps which coincide with them in any bounded domain and which are well defined at infinity. The fact that plane waves in all the space R^3 are not expressable as global knots is not a matter of concern, since a plane wave extending to all three-space is not in fact a physical solution since it requires an infinite amount of energy.

(iii) *Standing wave* given by

$$\begin{aligned}A_0&=&0,\quad A_1=A_{01}\cos k_1x\ \sin k_2y\ \sin k_3z\ \cos\omega t,\\ A_2&=&A_{02}\sin k_1x\ \cos k_2y\ \sin k_3z\ \cos\omega t,\\ A_3&=&A_{03}\sin k_1x\ \sin k_2y\ \cos k_3z\ \cos\omega t,\end{aligned}\tag{16}$$

which expresses one mode of a cubic cavity. The scalars which give this field can be taken as

$$\phi=\sqrt{\frac{1-p}{p}}e^{i2\pi q},\quad \theta=\sqrt{\frac{1-v}{v}}e^{i2\pi u},\tag{17}$$

where the Clebsh variables are equal to

$$p=\frac{1}{2}(1+\sin k_1x\ \sin k_2y\ \sin k_3z\cos\omega t),\quad q=\sum_{i=1}^{3}\frac{2A_{0i}}{\sqrt{a}k_i}\log|\sin k_ix_i|,$$

$$v=\frac{1}{2}(1+\cos k_1x\ \cos k_2y\ \cos k_3z\sin\omega t),\quad u=\sum_{i=1}^{3}\frac{2(\mathbf{k}\times\mathbf{A}_0)_i}{\sqrt{a}\omega k_i}\log|\sin k_ix_i|.$$

Note that the scalar fields ϕ and θ are not well defined in the planes $k_1x=n_1\pi$, $k_2y=n_2\pi$ and $k_3z=n_3\pi$, the n_i being integers, where q and u diverge. But there are scalars $\phi_{n_1n_2n_3}$, well defined and smooth in the finite domains $n_1\pi<k_1x<(n_1+1)\pi$, $n_2\pi<k_2y<(n_2+1)\pi$, $n_3\pi<k_3z<(n_3+1)\pi$, which generate the fields in each one of them. However, the electric and magnetic fields can not be produced by a pair of smooth maps $S^3\mapsto S^2$. As we said before, the fields can be obtained by patching together knots defined in bounded domains. Looking locally, this electromagnetic wave coincides with a knot around any point (except for a zero measure set), but there is no knot which coincides with it throught all the space R^3.

3 A property of the helicities of a radiation field

As was shown in references [3, 14, 15] (see also [16]), the electromagnetic helicity of an electromagnetic field, defined as the sum $\mathcal{H} = h_m + h_e$ satisfies

$$\mathcal{H} = 2\hbar c(N_R - N_L) \ , \tag{18}$$

where $N_R = \int d^3k \bar{a}_R a_R$, $N_L = \int d^3k \bar{a}_L a_L$, and $a_R(\mathbf{k})$, $a_L(\mathbf{k})$ being here Fourier transforms of the classical vector potential $\mathbf{A}$ ($\bar{a}_R$, $\bar{a}_L$ being their complex conjugates), *i. e.* the c-number fields which are interpreted in QED as annihilation operators for right and left polarization photons. It must be stressed that both the functions a_R, a_L and the right hand side of (18) are fully meaningful as classical quantities. Note that N_R and N_L are interpreted in QED as the numbers of right- and left-handed photons. Equation (18) is a remarkable relation between the wave and particle aspects of the field.

In this paper we consider only classical electromagnetic fields in empty space for which the energy and helicity integrals are both finite, so that their electric and magnetic vectors decrease faster than r^{-2} at infinity. Also a_R and a_L must be less singular than $\omega^{-3/2}$ (where $\omega = ck$) at $\omega = 0$ and decrease faster than ω^2 at $\omega = \infty$. We will prove now the following statement:

The magnetic and electric helicities of a radiation electromagnetic field in empty space are equal.

The proof is simple. By using the same method as in references [3, 14], it is easy to show that:

$$h_m - h_e = \hbar c \int d^3k \left[e^{-i\omega\tau} \left(a_L(\mathbf{k}) a_L(-\mathbf{k}) - a_R(\mathbf{k}) a_R(-\mathbf{k}) \right) + cc \right] \ , \tag{19}$$

where $\tau = 2t$. Defining now

$$F(\omega) = \hbar c \omega^2 \int d\Omega \left[a_L(\mathbf{k}) a_L(-\mathbf{k}) - a_R(\mathbf{k}) a_R(-\mathbf{k}) \right] \ ,$$

Ω being the solid angle and $F(\omega) = F(-\omega)$. Because of the stated behaviour of a_R and a_L, $F(\omega)$ is a square integrable function; it is clear then that

$$h_m - h_e = f(\tau) + \bar{f}(\tau) \ , \tag{20}$$

$f(\tau)$ being the Fourier transform of $F(\omega)$.

Taking now time derivatives, it is straightforward to see that

$$\frac{d}{dt}(h_m - h_e) = -4c \int_{R^3} \mathbf{E} \cdot \mathbf{B} \; d^3 r \; , \tag{21}$$

except for a vanishing surface integral. If the field is of radiation type, $\mathbf{E} \cdot \mathbf{B} = 0$, then the left hand side of (21) vanishes, which implies that the difference $h_m - h_e$ is a constant of the motion. Because of (20), the real part of $f(\tau)$ has a constant value, which can only be zero because it is a square integrable function. This means that $h_m = h_e$.

This property is valid for any electromagnetic field that satisfies that the integral in R^3 of $\mathbf{E} \cdot \mathbf{B}$ is zero. In particular it is so for any electromagnetic knot, what implies from (4) that $H(\phi) = H(\theta) = n$. Then (18) takes the form

$$\mathcal{H} = 2an = 2\hbar c (N_R - N_L) \; . \tag{22}$$

As a consequence, if one takes a definite value for the normalizing constant a, the knots have a curious mathematical property (in addition to any standard electromagnetic radiation field coinciding locally with a knot): their helicities verify the equations

$$h_m = h_e = na, \quad N_R - N_L = n \left(\frac{a}{\hbar c} \right) . \tag{23}$$

In other words, the knots can be classified in homotopy classes, since both the helicities and the classical quantity $N_R - N_L$ take only discrete values, which are integer multiples of a and $a/\hbar c$, respectively. Note that the dimensions of a are action times velocity.

The simplest thing to do now is taking $a = \hbar c$, since one has then $h_m = h_e = n\hbar c$, $N_R - N_L = n$, so that *the difference* $N_R - N_L$ *coincides with the common value of the electric and magnetic helicities.* This could be expressed by saying that, with this value of a, the knots are classical fields with the right normalization to be the classical limit of the quantum theory (as $N_R - N_L$ is an integer). As any standard radiation field coincides locally with a knot, this new property of the classical Maxwell equations implies that they are locally equivalent to a topological theory.

Note in this connection that, if one multiplies the phases of ϕ, θ by the integer j, the fields $\mathbf{B}$ and $\mathbf{E}$ are multiplied by j; and the energy and the helicities by j^2. In this way, we can define knots with dynamical quantities as big as desired, even with a small value of a, which plays the role of a unit of helicity, in spite of the knots being given by classical fields.

Let us define a *composed electromagnetic knot* as the sum of two knots; because of the Darboux theorem, any electromagnetic field is locally a composed knot. The electromagnetic helicity for a composed knot, sum of the knots (ϕ_1, θ_1) and (ϕ_2, θ_2), is:

$$\mathcal{H} = 2a\left(H(\phi_1) + H(\phi_2)\right) + 2\int_{R^3} \left(\mathbf{A}_1 \cdot \mathbf{B}_2 + \mathbf{C}_1 \cdot \mathbf{E}_2\right)\ d^3r\ . \tag{24}$$

In [3], a family of knots was constructed by imposing their defining property at $t = 0$. It turns out that, although all of them are composed knots, only those for which the magnetic and the electric linking numbers are equal at $t = 0$ are knots in the strict sense (as it is to be expected from the property just proved). This will be shown now by giving the explicit time dependent expressions of the corresponding scalars. The scalars at $t = 0$ were given by:

$$\begin{aligned} \phi^{(n)}(t=0) &= \phi_H^{(n)} = \left(\frac{2X + i2Y}{2Z + i(R^2-1)}\right)^{(n)} \\ \theta^{(m)}(t=0) &= \theta_H^{(m)} = \left(\frac{2Y + i2Z}{2X + i(R^2-1)}\right)^{(m)} \end{aligned} \tag{25}$$

where $(X, Y, Z) = \lambda(x, y, z)$, λ being any constant with inverse lenght dimension, and ϕ_H, θ_H are Hopf maps with unit Hopf index. The notation $\phi^{(n)}$ means to leave the modulus of the complex number ϕ unchanged and multiplying its phase by the integer n. It is easy to see [3] that:

$$\begin{aligned} H(\phi^{(n)}) &= n^2 H(\phi) = n^2 \\ H(\theta^{(m)}) &= m^2 H(\theta) = m^2 \end{aligned} \tag{26}$$

Taking as Cauchy data the magnetic and electric fields derived from $\phi^{(n)}$, $\theta^{(m)}$, respectively, we obtain the time dependent electromagnetic field (following the method of reference [1])

$$\begin{aligned} \mathbf{B}_n(\mathbf{r}, t) &= \frac{\sqrt{a}\lambda^2}{\pi(A^2+T^2)^3}\left(Q\mathbf{H}_1 + P\mathbf{H}_2\right) \\ \mathbf{E}_m(\mathbf{r}, t) &= \frac{-\sqrt{a}\lambda^2}{\pi(A^2+T^2)^3}\left(P\mathbf{H}_1 - Q\mathbf{H}_2\right) \end{aligned} \tag{27}$$

with $T = \lambda ct$, where:

$$Q = A(A^2 - 3T^2) \ , \ \ P = T(T^2 - 3A^2) \ , \ \ A = \frac{R^2 - T^2 + 1}{2} \ , \qquad (28)$$

the vectors $\mathbf{H}_1$ and $\mathbf{H}_2$ being

$$\begin{aligned} \mathbf{H}_1 &= n\left(Y - XZ \ , \ -X - YZ \ , \ \frac{X^2 + Y^2 - Z^2 + T^2 - 1}{2} \right) \\ &+ \ mT\,(\,1\ ,\ -Z\ ,\ Y\,) \\ \mathbf{H}_2 &= m\left(\frac{X^2 - Y^2 - Z^2 - T^2 + 1}{2} \ , \ -Z + XY \ , \ Y + XZ \right) \\ &+ \ nT\,(\,-Y\ ,\ X\ ,\ 1\,) \end{aligned} \qquad (29)$$

It is easy to show that, if $n = m$, this solution is a knot. More precisely, it is a simple, if cumbersome, exercise to show that (27) are obtained from the scalars

$$\begin{aligned} \phi^{(n)}(t) &= \left(\frac{(AX - TZ) + i(AY + T(A-1))}{(AZ + TX) + i(A(A-1) - TY)} \right)^{(n)} \\ \theta^{(n)}(t) &= \left(\frac{(AY + T(A-1)) + i(AZ + TX)}{(AX - TZ) + i(A(A-1) - TY)} \right)^{(n)} \end{aligned} \qquad (30)$$

each one having Hopf index equal to n^2. In this way, we have obtained the explicit time dependent expressions of representative knots of the homotopy classes whith Hopf index n^2. Now, changing the sign of (X, Y, Z, T) in (30), we obtain the homotopy classes with Hopf index $-n^2$ (the same can be achieved also by other changes).

Let us take now the electromagnetic field (27) with $n^2 \neq m^2$. It can be shown, just with some algebra, that it is then the sum of two knots, with scalars (ϕ_1, θ_1) and (ϕ_2, θ_2), each one with a common value for the electric and magnetic helicities. Let n_1, n_2 (both integers), α, β verify the system:

$$\begin{aligned} n_1 + n_2 &= m \ , \\ n_1 \cos\alpha + n_2 \cos\beta &= n \ , \\ n_1 \sin\alpha + n_2 \sin\beta &= 0 \ . \end{aligned} \qquad (31)$$

This system has the following solution for each couple (m, n). If $n^2 > m^2$, take an integer k such that $n^2 \leq k^2$ and having the same parity as m. Then, we can take $n_1 = (m-k)/2$, $n_2 = (m+k)/2$, $\cos\alpha = (n^2 - km)/(nm - nk)$, $\cos\beta = (n^2 + km)/(nm + nk)$. If, in the other case, $n^2 < m^2$, the same solution is valid with $k^2 \leq n^2$. The only situation in which the system (31) has no solution is the case $n = 0$ and m odd, but we can deal with it changing the roles of n and m (equivalently, changing those of the two scalars).

Then ϕ_1 and θ_1 are given by:

$$
\begin{aligned}
\phi_1(t) &= \left(\frac{(AX - TZ)\cos\alpha - (A(A-1) - TY)\sin\alpha + i(AY + T(A-1))}{(AZ + TX) + i(A(A-1) - TY)\cos\alpha + i(AX - TZ)\sin\alpha} \right)^{(n_1)} \\
\theta_1(t) &= \left(\frac{(AY + T(A-1)) + i(AZ + TX)}{(AX - TZ) + i(A(A-1) - TY)} \right)^{(n_1)}
\end{aligned}
\qquad (32)
$$

and ϕ_2 and θ_2 have the same form as (32), with the changes $n_1 \leftrightarrow n_2$ and $\alpha \leftrightarrow \beta$. From these expressions it is easy to see that $H(\phi_1) = H(\theta_1) = n_1^2$, and $H(\phi_2) = H(\theta_2) = n_2^2$. To show that these two are knots and that their sum is the composed knot given by (27), the best is writing them explicitely. For ϕ_1, θ_1, we have the knot:

$$
\begin{aligned}
\mathbf{B}_{n_1}(\mathbf{r}, t) &= \frac{\sqrt{a}\lambda^2 n_1}{\pi(A^2 + T^2)^3} \left(Q\mathbf{F}_1 + P\mathbf{F}_2 \right) \\
\mathbf{E}_{n_1}(\mathbf{r}, t) &= \frac{-\sqrt{a}\lambda^2 n_1}{\pi(A^2 + T^2)^3} \left(P\mathbf{F}_1 - Q\mathbf{F}_2 \right)
\end{aligned}
\qquad (33)
$$

where P, Q, A are given by (28) and:

$$
\begin{aligned}
\mathbf{F}_1 &= \cos\alpha \left(Y - XZ \,,\, -X - YZ \,,\, \frac{X^2 + Y^2 - Z^2 + T^2 - 1}{2} \right) \\
&+ \sin\alpha \left(-Z - XY \,,\, \frac{X^2 - Y^2 + Z^2 + T^2 - 1}{2} \,,\, X - YZ \right) \\
&+ T\,(\, 1 \,,\, -Z \,,\, Y \,) \\
\mathbf{F}_2 &= \left(\frac{X^2 - Y^2 - Z^2 - T^2 + 1}{2} \,,\, -Z + XY \,,\, Y + XZ \right) \\
&+ T\cos\alpha\,(\, -Y \,,\, X \,,\, 1 \,) \\
&+ T\sin\alpha\,(\, Z \,,\, 1 \,,\, -X \,)
\end{aligned}
\qquad (34)
$$

and a similar form for $\mathbf{E}_{n_2}$ and $\mathbf{B}_{n_2}$. The sums $\mathbf{B}_{n_1} + \mathbf{B}_{n_2}$ and $\mathbf{E}_{n_1} + \mathbf{E}_{n_2}$ are equal to (27) if (31) holds true.

4 A topological model of electromagnetism

The preceding considerations show that a model for which the radiation fields are electromagnetic knots is locally equivalent to the standard Maxwell theory [2]. The reason is that, as explained in Section 2, any standard field can be written as the sum of two radiation fields, which implies that is locally equal to the sum of two knots. Of course, they would be non equivalent from the global point of view. This topological model can be formalized by means of a variational principle as follows. Let us take four scalars ϕ_k, θ_k, $k = 1, 2$, as fundamental fields and define

$$\mathcal{F} = -\sqrt{a}\,(\phi_1^*\sigma + \phi_2^*\sigma)\ ,\ \ *\mathcal{F} = \sqrt{a}\,(\theta_1^*\sigma + \theta_2^*\sigma)\ , \tag{35}$$

where the star superscripts indicate here pull-back of the area 2-form σ in S^2 to the Minkowski space-time $M = R^3 \times R$ by the corresponding map. It seems natural to take as action integral

$$\mathcal{S} = \frac{-1}{4}\int (\mathcal{F}(\phi_k) \wedge *\mathcal{F}(\phi_k) + *\mathcal{F}(\theta_k) \wedge \mathcal{F}(\theta_k))\ , \tag{36}$$

which coincides with the usual form $-1/2(\mathcal{F} \wedge *\mathcal{F})$. The duality conditions

$$-*(\phi_k^*\sigma) = \theta_k^*\sigma\ ,\ \ k = 1, 2\ , \tag{37}$$

must be imposed by means of the Lagrange multipliers method. It is very easy to show that the Euler-Lagrange equations are:

$$d\mathcal{F} = 0\ ,\ \ d{*}\mathcal{F} = 0\ , \tag{38}$$

since the duality conditions do not contribute to these equations, what means that they are naturally conserved under time evolution. This shows that a topological model locally equivalent to Maxwell theory is possible.

In standard Maxwell theory, the Cauchy data are the eight functions A_μ, $\partial_0 A_\mu$, and there is gauge invariance. In this topological model, they are the four complex functions $\phi_k(\mathbf{r}, 0)$, $\theta_k(\mathbf{r}, 0)$, that is eight real functions, constrained by the two conditions $(\nabla\bar{\phi}_k \times \nabla\phi_k) \cdot (\nabla\bar{\theta}_k \times \nabla\theta_k) = 0$, in order that the level curves of ϕ_k will be orthogonal to those of θ_k. It is not necessary to prescribe the time derivatives $\partial_0\phi_k$, $\partial_0\theta_k$ since they are determined by the duality conditions (37).

The model thus constructed looks linear because the fields **B**, **E** obey the linear Maxwell equations in empty space. However, it can not be really linear since it embodies topological constants of the motion. Given a knot **B**, **E**, the field $\lambda\mathbf{B}$, $\lambda\mathbf{E}$ is only a solution of the model if λ is an integer, a case in which it is also a knot; in spite of that any standard electromagnetic field is locally equal to a composed knot of the model. The difference is in the way the fields behave around the infinity. What happens is that the composed knots form a nonlinear subset of the set of solutions of Maxwell equations, which can be considered as linear for local purposes. This is a subtle way of being nonlinear and can be called hidden nonlinearity.

5 Summary and conclusions

We have proved (i) that any standard electromagnetic field of radiation type is locally equal to an electromagnetic knot, and (ii) that the magnetic and electric helicities of a knot, and of any standard electromagnetic field of radiation type, are equal. Consequently, a topological model can be constructed which is locally equivalent to Maxwell theory.

As a summary, it can be said that electromagnetic knots are interesting objects worth of further research.

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