

Topological Quantization of the Magnetic Flux

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The quantization of the magnetic flux in superconducting rings is studied in the frame of a topological model of electromagnetism that gives a topological formulation of electric charge quantization. It turns out that the model also embodies a topological mechanism for the quantization of the magnetic flux with the same relation between the fundamental units of magnetic charge and flux as there is between the Dirac monopole and the fluxoid.

KEY WORDS: superconductivity; magnetic flux quantization; topological fields; topological quantization.

1. THE TOPOLOGICAL MODEL OF ELECTROMAGNETISM

This section summarizes the basic elements of the a topological model of electromagnetism (TME from now on) previously proposed by one of us,⁽¹⁾ which is locally equivalent to Maxwell's standard theory but implies furthermore some topological quantization conditions with intriguing physical implications^{(2)–(5)} (⁽³⁾ is a review of the results obtained up to 2001). The TME makes use of two fundamental complex scalar fields (ϕ, θ) , their level curves being the magnetic and electric lines, respectively, so that each one of these lines is labelled by a particular value of the corresponding scalar. It turns out that the set of magnetic and electric lines has very curious and interesting topological properties.

The two scalars are assumed to have only one value at infinity, which is equivalent to compactifying the three-space into the sphere S^3 . This implies that they can be interpreted (via stereographic projection) as two maps $S^3 \rightarrow S^2$, which can be classified in homotopy classes and, as such,

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characterized by the value of the Hopf index n . It can be shown that the two scalars have the same Hopf index and that the magnetic (resp. electric) lines are generically linked with the same Gaussian linking number ℓ . If μ is the multiplicity of the level curves (i.e., the number of different magnetic (resp. electric) lines that have the same label ϕ (resp. θ)), then $n = \ell\mu^2$; the Hopf index can thus be interpreted as a generalized linking number if we define a line as a level curve with μ disjoint components.

An important feature of the model is that the Faraday 2-form $\mathcal{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ and its dual $*\mathcal{F} = \frac{1}{2}*F_{\mu\nu}dx^\mu \wedge dx^\nu$ are proportional to the two pull-backs of σ , the area 2-form in S^2 , by ϕ and θ , i.e.,

$$\mathcal{F} = -\frac{\sqrt{a}}{c}\phi^*\sigma, \quad *\mathcal{F} = \frac{\sqrt{a}}{c}\theta^*\sigma, \quad (1)$$

where \sqrt{a} is a constant with dimensions of electric flux and a value of $\sqrt{a} = \sqrt{\hbar c/\epsilon_0}$ in SI units (\hbar, c, ϵ_0 being the Planck's constant, the light velocity and the vacuum permittivity). Natural units will be used here, so that $\sqrt{a}/c = 1$. Consequently, the two maps are dual to one another in the sense that

$$*(\phi^*\sigma) = -\theta^*\sigma, \quad (2)$$

where $*$ is the Hodge or duality operator. Curiously enough, the existence of two maps satisfying (2) guarantees that both \mathcal{F} and $*\mathcal{F}$ obey Maxwell's equations in empty space without the need for any other requirement. We will note $\mathcal{F} \equiv (\mathbf{E}, \mathbf{B})$, $*\mathcal{F} \equiv (-\mathbf{B}, \mathbf{E})$.

The electromagnetic fields having the form (1) are called "electromagnetic knots". They are radiation fields, i.e., they obey the condition $\mathbf{E} \cdot \mathbf{B} = 0$. It must be stressed here that, because of the Darboux theorem, any electromagnetic field in empty space can be expressed locally as the sum of two radiation fields.

The pair of scalar fields (ϕ, θ) generate a Faraday 2-form and its dual which can be expressed as

$$\mathcal{F} = \frac{\sqrt{a}}{c} ds \wedge dp, \quad \text{with } p = 1/(1 + |\phi|^2), \quad s = \arg(\phi)/2\pi, \quad (3)$$

$$*\mathcal{F} = \frac{\sqrt{a}}{c} dv \wedge du, \quad \text{with } v = 1/(1 + |\theta|^2), \quad u = \arg(\theta)/2\pi, \quad (4)$$

so that $\phi = \sqrt{(1-p)/p} e^{i2\pi s}$ and $\theta = \sqrt{(1-v)/v} e^{i2\pi u}$.

This implies that the magnetic and electric fields have the form

$$\mathbf{B} = \frac{\sqrt{a}}{c} \nabla p \times \nabla s = \frac{\sqrt{a}}{c} (\partial_0 u \nabla v - \partial_0 v \nabla u), \quad (5)$$

$$\mathbf{E} = \sqrt{a} \nabla u \times \nabla v = \sqrt{a} (\partial_0 s \nabla p - \partial_0 p \nabla s). \quad (6)$$

The quantities (p, s) and (v, u) are called the *Clebsch variables* of the fields \mathbf{B} and \mathbf{E} , respectively (and of the scalars ϕ and θ as well). Note that ϕ and θ are not uniquely determined by the magnetic and electric fields. Indeed, a different pair defines the same fields \mathbf{E} , \mathbf{B} if the corresponding Clebsch variables (P, S) , (V, U) can be obtained through a canonical transformation $(p, s) \rightarrow (P, S)$ or $(v, u) \rightarrow (V, U)$. However, the canonical transformation must satisfy two conditions: (i) $0 \leq P, V \leq 1$ and (ii) S, U must be the arguments of complex functions in units of 2π , i.e., they can be multi-valued but their change along a closed curve must be an integer. Changes in the Clebsch variables will be introduced later in the article.

2. THREE QUANTIZATION CONDITIONS

As stated before, the TME is locally equivalent to Maxwell's standard theory.^(2,5) However, their differences from the global point of view are quite interesting, as seen in the following three topological quantizations:

(i) In the TME, the electric charge of any point particle must necessarily be equal to an integer multiple of the fundamental value $e_0 = \sqrt{\hbar c \epsilon_0}$, in SI units (or $e_0 = 1$ in natural units). Furthermore, if a charge has m fundamental units $q = m e_0$, then m is a topological index equal to the degree of the map $\theta' : \Sigma \rightarrow S^2$, i.e., the restriction of θ to any closed surface Σ enclosing the charge. It follows that, generically, there are exactly m lines converging to or diverging from a charge $m e_0$ that have any prescribed complex value of θ as their common label. Note that $e_0 = 3.3 e = 5.29 \times 10^{-19}$ C, where e is the electron charge⁽²⁾. The same mechanism applies to any hypothetical magnetic charge, so that the model has room for monopoles if and only if their magnetic charges are integer multiples of $g_0 = e_0/c = \sqrt{\hbar \epsilon_0/c}$. In natural units, the electric and magnetic fundamental charges of the TME are equal $e_0 = g_0 = 1$, the corresponding fine structure constant is $\alpha_0 = 1/4\pi$, $e = 0.3028$ and the magnetic charge of the Dirac monopole $g = 2\pi/e = 20.75$. It must be stressed here that Maxwell's theory embodies no charge quantization.

As the TME is classical, e_0 and α_0 must be interpreted as bare values. In Section 8 of Ref. 3, it is argued that this interpretation is plausible since the quantum vacuum is dielectric but paramagnetic, so that it

changes the electron charge from e_0 to e ($< e_0$) and the monopole charge from $g_0 = e_0/c$ to g ($> g_0$). Intriguingly, it turns out that α_0 is close to the estimated value of α_s at the unification scale. This suggests that the topological model gives a theory of bare electromagnetism (i.e., without the effect of the quantum vacuum) or for high energies (at the unification scale), since in this limit one can argue that the bare charges interact directly. The symmetry implied by the TME between electricity and magnetism would be broken by the effect of the quantum vacuum.

(ii) The electromagnetic helicity \mathcal{H} is also quantized. In natural units,

$$\mathcal{H} = \frac{1}{2} \int_{R^3} (\mathbf{A} \cdot \mathbf{B} + \mathbf{C} \cdot \mathbf{E}) d^3r = n, \quad (7)$$

where $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = \nabla \times \mathbf{C}$, the integer n being equal to the common value of the Hopf indices of ϕ and θ (in physical SI units, the term $\mathbf{C} \cdot \mathbf{E}$ in (7) would be divided by c^2 , the right-hand side being equal to $n\hbar/c\epsilon_0$). Note that $\mathcal{H} = N_R - N_L$, where N_R and N_L are the classical expressions of the number of right- and left-handed photons contained in the field (i.e., $\mathcal{H} = N_R - N_L = \int d^3k (\bar{a}_R a_R - \bar{a}_L a_L)$, $a_R(\mathbf{k})$, $a_L(\mathbf{k})$ being Fourier transforms of A_μ in the classical theory, but are the creation and annihilation operators in the quantum version). This implies that

$$n = N_R - N_L, \quad (8)$$

which reveals a curious relation between the Hopf index (i.e., the generalized linking number, which is equal to the electromagnetic helicity) of the classical field and the classical limit of the difference $N_R - N_L$. This difference has a clear topological meaning, and is attractive from the intuitive physical point of view.

(iii) The topology of the model also implies the quantization of the energy of the electromagnetic field in a cavity. More precisely, it predicts that its energy \mathcal{E} in a cubic cavity will obey the relation

$$\mathcal{E} = n\omega, \quad (9)$$

where $n = d/4$, and d is an integer equal to the degree of a certain map between two orbifolds. This rule is different from yet very similar to the Planck–Einstein law. The two rules are identical when d is a multiple of 4.

Next we will show that the TME predicts the quantization of the magnetic flux of a superconducting ring, which in standard theory is always an integer multiple of half the fluxoid $g/2$, where g is the Dirac monopole. In the case of this topological model, the unit of magnetic flux

is $g_0/2$, where $g_0 = e_0/c$ is the fundamental magnetic charge of the model, equal to $g_0 = \sqrt{\hbar\epsilon_0}/c$ in SI physical units (or, equivalently, $g_0 = 1$ in natural units). This means that the relationship between the fundamental charge and the fluxoid in this model is the same as in standard theory.

3. FLUX QUANTIZATION IN AN INFINITE SOLENOID

Let us consider an infinite perfect solenoid around the z -axis. The adjective *perfect* means that no flux escapes through the coils. Since this can only happen in a superconducting ring, “perfect solenoid” and “superconducting solenoid” are synonymous in this work. So, the magnetic field vanishes outside and is constant and equal to $\mathbf{B} = B\mathbf{e}_z$ inside the ring, where \mathbf{e}_z is unit vector along the axis. In the TME, this field strength corresponds to the scalar ϕ (which gives a map $S^3 \mapsto S^2$).⁽³⁾ Given the configuration of the magnetic lines of this solenoid, it is impossible for ϕ to be regular in the entire sphere S^3 . However, we may consider the 3-space as $S^2 \times R$ and require that ϕ be regular for the induced map $S^2 \mapsto S^2$, the first S^2 being the plane (x, y) , and the second the complex plane, both completed with the point at infinity). If $\phi = |\phi| \exp(2\pi i s)$ and $p = 1/(1 + |\phi|^2)$, then

$$\mathbf{B} = \nabla p \times \nabla s. \quad (10)$$

Since $\mathbf{B} = 0$ outside the solenoid, p and s can not be independent functions there. This could happen in three different ways: (i) $s = f(p)$, f being a nontrivial function, or (ii) $s = s_0 = \text{constant}$, (iii) or $p = p_0 = \text{constant}$.

(i) If $s = f(p)$, we can substitute s with $s - f(p)$. This is a canonical transformation of the variables s, p which does not affect the value of \mathbf{B} in view of Eq. (10). The new expression of ϕ is real outside, but complex inside the solenoid in general. Consequently, the magnetic flux across a section of the solenoid is topologically quantized, and is equal to the area of the set $\phi(S)$ in the sphere S^2 , where S is any surface that completely cuts the solenoid and is bordered by a circuit outside it. In fact, this flux must necessarily be $\text{Flux} = n/2$, because any curve contained in a great circle of a sphere encircles an integral multiple of semi-spheres. Or, considering the stereographic projection, if the curve is contained in the real axis it encircles an integral number of semi-planes.

(ii) If s is constant, the situation is similar to and gives the same flux quantization as (i) (outside the solenoid, ϕ takes values also in a great circle of S^2). Hence $\text{Flux} = n/2$.

This is interesting, because it means that when either (i) or (ii) is true the flux in the solenoid is necessarily quantized, and the fundamental fluxoid is $g_0/2$, just as in standard theory the real fluxoid is half the Dirac monopole $g/2 = 10.37$.

(iii) Let $p = p_0$ outside, and s variable—otherwise we would have case (ii). Then the scalar would be

$$\phi = \sqrt{\frac{1-p_0}{p_0}} e^{i2\pi s(r, \varphi)}, \quad (11)$$

where $r = (x^2 + y^2)^{1/2}$ and φ is the azimuth. Moreover,

$$\int_0^{2\pi} \frac{\partial s}{\partial \varphi} d\varphi = m, \quad (12)$$

i.e., an integer. A simple example of such a function is $s = m\varphi/2\pi$.

In order for ϕ to be a regular map of the plane xy on the complete complex plane with s a non-constant function of φ , either $p_0 = 0$ (so that $\phi = \infty$) or $p_0 = 1$ (or $\phi = 0$). Otherwise ϕ would not be well defined at infinity in this plane. In both cases, it turns out that $\text{Flux} = n/2$.

Conclusion: In the topological model of electromagnetism, the magnetic flux in an infinite perfect solenoid is always an semi-integral multiple of the fundamental magnetic flux $\sqrt{a}/c = 1$, so that

$$\text{Flux} = \frac{n}{2}. \quad (13)$$

4. FLUX QUANTIZATION IN A FINITE SOLENOID I

Let us consider now the case of a superconducting ring (i.e., of a perfect but finite solenoid), limited by the planes $z = -L/2$ and $z = L/2$ and two cylinders around de z axis with radii r_0 and $r_0 + \delta r_0$, although the particular values of these two radii are not relevant. Since the magnetic field does not enter inside the superconductor, $\mathbf{B} = 0$ inside it.

If the superconductor is infinitely thick (i.e., $\delta r_0 = \infty$), the topology of the problem is the same as before, and all the results are also the same. In the realistic case in which δr_0 is finite, there are also three cases, as in Section 3, just after Eq. (10), but substituting “within the superconducting material” for “outside the solenoid”. It is evident that the result would be the same in the two first cases (i) and (ii). However, it is not clear at first glance that the same could be said of case (iii).

In order to study the third case, let us apply these ideas to the quantization of the magnetic flux across a superconducting ring⁽⁶⁾ using

standard theory. In this case the wave function can be treated as a classical macroscopic field $\psi = \sqrt{\rho} e^{i\vartheta}$, the following equation being satisfied

$$\hbar \nabla \vartheta = Q \mathbf{A}, \tag{14}$$

where $Q = 2e$ is the charge of a Cooper pair of electrons. The flux is thus

$$\text{Flux} = \oint \mathbf{A} \cdot d\mathbf{s} = \frac{2\pi n'}{Q}, \tag{15}$$

n' being an integer. We see that the fundamental unit of flux is $2\pi/Q$. Note that the phase shift in an Bohm–Aharonov experiment is $e\text{Flux}/\hbar$.

Let us take the case of a finite superconducting ring of cylindrical shape, as explained before. The interior magnetic field in the central plane $z = 0$ is

$$\mathbf{B} = B(r) \mathbf{e}_z, \tag{16}$$

where r is the radial coordinate, and the magnetic flux is

$$\text{Flux} = \int_{C_0} B(r) r dr d\varphi, \tag{17}$$

where C_0 is the circle of radius r_0 . Because of the symmetry of the problem, we can take a scalar $\phi(r, \varphi)$, with $p = 1/(1+|\phi|^2)$ and $s = \arg(\phi)/2\pi$, such that

$$\mathbf{B} = \frac{1}{r} \left(\frac{\partial p}{\partial r} \frac{\partial s}{\partial \varphi} - \frac{\partial p}{\partial \varphi} \frac{\partial s}{\partial r} \right) \mathbf{u}_z. \tag{18}$$

It is convenient to define the dimensionless radial coordinate as

$$R = \frac{r}{r_0}, \tag{19}$$

so that, in each plane (R, φ) , ϕ can be taken as a map $\phi: C_1 \rightarrow S^2$, where C_1 is the circle with $R = 1$, and

$$\mathbf{B} = \frac{1}{r_0^2} \frac{1}{R} \left(\frac{\partial p}{\partial R} \frac{\partial s}{\partial \varphi} - \frac{\partial p}{\partial \varphi} \frac{\partial s}{\partial R} \right) \mathbf{u}_z. \tag{20}$$

The magnetic flux across the superconductor turns out to be

$$\text{Flux} = \int_{C_1} \left(\frac{\partial p}{\partial R} \frac{\partial s}{\partial \varphi} - \frac{\partial p}{\partial \varphi} \frac{\partial s}{\partial R} \right) dR d\varphi. \tag{21}$$

The quantity between brackets in (21) is the Jacobian of the change of variables $(p, s) \rightarrow (R, \varphi)$, so that

$$\text{Flux} = \int_{\phi(C_1)} dp ds, \quad (22)$$

where $\phi(C_1)$ is the image in S^2 of the unit circle C_1 .

The magnetic field in the superconductor satisfies a phenomenological equation in the transition layer in which the magnetic field goes to zero. This is the second London equation

$$\mathbf{A} = -\lambda^2 \nabla \times \mathbf{B}, \quad (23)$$

where $\mathbf{A}(\mathbf{r})$ is in the Coulomb gauge and λ is the penetration length of the magnetic field inside the superconductor material (in practice, λ is about 10 \AA , much shorter than the inner radius of the superconductor ring r_0).

5. FLUX QUANTIZATION IN A FINITE SOLENOID II

Let us write $\mathcal{F} = d\mathcal{A}$, where $\mathcal{A} = A_\mu dx^\mu$. In the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$), the vector potential $\mathbf{A}(\mathbf{r})$ is purely azimuthal and its modulus depends only on the radial coordinate r . From (5) and (18) the vector potential can be written as

$$\mathbf{A} = A(r) \mathbf{u}_\varphi = \frac{1}{r} p \frac{ds}{d\varphi} \mathbf{u}_\varphi. \quad (24)$$

It follows that $s = s(\varphi)$ (otherwise \mathbf{A} would not be azimuthal) and $p = p(r)$, since $p(ds/d\varphi)$ must be regular on the ring axis. Furthermore $\int_0^{2\pi} A_\varphi r d\varphi$ must be independent of r inside the superconductor, from which

$$p = p_0, \quad s = n \frac{\varphi}{2\pi}. \quad (25)$$

Inserting Eq. (24) into the London Eq. (23), we obtain the following ordinary differential equation for $p(r)$,

$$\lambda^2 \left(\frac{d^2 p}{dr^2} - \frac{1}{r} \frac{dp}{dr} \right) - p = 0. \quad (26)$$

Up to the first order in λ/r_0 , we can neglect the first term in (26) to obtain

$$p(r) = 0, \quad r \geq r_0, \quad (27)$$

which characterizes p inside the superconductor. Since the Clebsch variable p has to be continuous and constant inside the superconductor, with a value $p = p_0$, we obtain $p_0 = 0$, i.e., $\phi = \infty$.

In the TME, if an electromagnetic field is generated by the scalar field ϕ and the Clebsch variables (p, s) , it is also generated by the scalar $1/\bar{\phi}$ and the Clebsch variables $(1-p, -s)$. In the latter case, Eq. (27) would be $1-p(r) = 0$, $r \geq r_0$, so that $p_0 = 1$ and $\phi = 0$ inside the superconductor.

Consequently, the value of the scalar field ϕ inside the superconductor is $\phi = \infty$ or $\phi = 0$. In both cases, the magnetic flux is

$$\text{Flux} = \int_{C_1} A(R) r_0 R d\varphi = \int_0^{2\pi} \frac{n}{2\pi} d\varphi = n. \quad (28)$$

If we consider the solutions given by the families (i) and (ii), we find that the magnetic flux is always quantized, since it is always an integral multiple of $1/2$, i.e., $\text{Flux} = n/2$.

This argument relies on London's equation. However, the same conclusion can be reached taking an alternate route. The radial derivative of p is in general discontinuous at r_0 . However, this irregularity in the map ϕ is eliminated if either $\phi = 0$ or $\phi = \infty$ inside the superconductor. Therefore, the requirement that the map is regular leads to the topological quantization of the flux, without taking into account the London equation.

6. CONCLUSION

As explained in previous papers, the topological model of electromagnetism (TME) presented in Ref. 1 embodies a topological mechanism for the quantization of the electric charge, the fundamental unit being 1 in natural units ($\approx 3.3e$). This work now shows that the TME also predicts that the magnetic flux is quantized, the fundamental flux unit being $1/2$ in natural units. Consequently, the relation between the fundamental magnetic flux and electric charge in the TME is the same as that between the Dirac monopole and the electron charge in standard theory.

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