

# The role of simulation methods in Economics

Alfonso Novales  
Departamento de Economía Cuantitativa  
Universidad Complutense  
June 2007

# INDEX

1. The issues
2. Why do we need to simulate?
3. What do we get out of model simulation?
4. Calibrating/estimating a theoretical model
  - What is calibration?
  - What is a reasonable model?
  - Limitations in calibration: parameters and exogenous stochastic processes
  - Calibration vs. estimation
5. Incorporating parameter uncertainty / Evaluation procedures for simulation models
6. Stability
7. Some statistical issues
8. The analysis of the transition between steady-states
9. Have we changed the type of questions we ask?
  - Cyclical vs. Countercyclical policy
  - Heterogeneous agents
  - The type of conclusions we reach
10. Summary

# 1. The issues

- Widespread use of dynamic, stochastic, model economies has led to the need of using numerical methods to characterize the properties of a given theoretical economy.
- There is still some misunderstanding regarding a correct use of numerical methods.
- The non-specialist has sometimes the impression that simulation is more a caprice of the researcher than a real need.
- Solution methods seem difficult to understand and replicate.
- Besides, assignment of numerical values to key structural parameters is thought from the outside to be an arbitrary decision, which totally conditions the results.
- Lastly, fundamental skepticism comes from considering whether results characterized by simulation can ever be compared to properties we learn about through a formal mathematical proof.

## 2. Why do we need to simulate?

- Formulate economic questions in dynamic, stochastic setups to avoid biased answers
  - Agents make decisions under uncertainty
  - The consequences of their actions are felt over a number of periods. They might start only after some period of time
  - And influence some other variables, that will feedback into the economy
  - Expectations formation mechanism is a crucial part of the structure of the model, and it decisively influences the characteristics of the model economy
- Need to solve stochastic control problems:
  - Pontryagin's principle, Bellman's dynamic optimization principle, Kushner's stochastic Lagrange multipliers
- Sargent (1979) *Macroeconomic Theory*, Academic Press
- Phelps et al. (1969), *Microeconomic Foundations of Employment and Inflation*, Norton
- Lucas (1983), *Studies in Business Cycle theory*, MIT Press
- Lucas and Sargent (1981), *Rational Expectations*, 2 vol.,

2. Why do we need to simulate?

# Diverse background

• Statistics / Econometrics

- Optimization theory
  - Kuhn-Tucker theorem
  - Optimal control
  - Stochastic control
- Numerical analysis  
(VERY IMPORTANT)
- Computer programming

- Dynamic microeconomics
  - Consumption choices
- Dynamic macroeconomics
  - Fiscal and monetary policy
  - Optimal policy design
- Growth Theory

# Example: Cagan's monetary model

## No optimization

$$\frac{M_t^d}{P_t} = \alpha_1' - \alpha_2({}_t\pi_{t+1}) \Rightarrow P_t = \frac{M_t}{\alpha_1} + b({}_tP_{t+1}), \quad \alpha_1 = \alpha_1' + \alpha_2, b = \frac{\alpha_2}{\alpha_1} < 1$$

$$\text{Adaptive expectations} \Rightarrow P_t = \frac{M_t}{\alpha_1} + b\lambda \sum_{i=0}^{\infty} (1-\lambda)^i P_{t-i}$$

$$\text{Rational expectations} \Rightarrow P_t = \frac{M_t}{\alpha_1} + \frac{1}{\alpha_1} \sum_{j=1}^{\infty} b^j E_t M_{t+j}$$

where the statement of Lucas' critique is already apparent:

It is agents' perceptions of future policy that matter, rather than the announcements of a given policy

The credibility of the policy maker is crucial

Possibility of unjustified market volatility

# Example: No uncertainty – single constraint

$$\begin{aligned} & \text{Max } \sum_1^T \beta^t \ln(C_t) \\ & \text{subject to: } \sum_1^T C_t \leq A \end{aligned}$$

$$\begin{aligned} L & \equiv \sum_1^T \beta^t \ln(C_t) - \lambda \left( \sum_1^T C_t - A \right) \\ \frac{\beta^t}{C_t} - \lambda & \leq 0 \text{ and } \left( \frac{\beta^t}{C_t} - \lambda \right) C_t = 0 \\ \sum_1^T C_t & \leq A \text{ and } \left( \sum_1^T C_t - A \right) \lambda = 0 \end{aligned}$$

$$C_t = K \beta^t \Rightarrow K (\beta + \beta^2 + \dots + \beta^T) = A \Rightarrow K = A \frac{1-\beta}{\beta} \frac{1}{1-\beta^T}$$

$$\text{Special case: } T = 2 \Rightarrow C_1 = A \frac{1}{1+\beta}; C_2 = A \frac{\beta}{1+\beta}$$

2. Why do we need to simulate?

# Same example: Bellman's principle

$$\begin{aligned} & \text{Max} \sum_1^T \beta^t \ln(C_t) \\ & \text{subject to: } \sum_1^T C_t \leq A \end{aligned}$$

First solve:

$$\begin{aligned} & \text{Max} \ln(C_2) \\ & \text{subject to: } C_2 \leq A - C_1 \end{aligned}$$

that leads to:  $C_2 = A - C_1$

Then, solve:

$$\begin{aligned} & \underbrace{\text{Max}}_{C_1} \ln(C_1) + \beta \ln(C_2) \\ & \text{subject to: } C_1 \leq A \end{aligned}$$

And taking into account the optimal rule at t=2:

$$C_2 = A - C_1$$

That is:

$$\begin{aligned} & \underbrace{\text{Max}}_{C_1} \ln(C_1) + \beta \ln(A - C_1) \\ & \text{subject to: } C_1 \leq A \end{aligned}$$

Leading to the same solution as before

# Example: No uncertainty – single constraint

$$\text{Max} \sum_1^T \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma}$$

subject to:  $\sum_1^T C_t \leq A$

$$L \equiv \sum_1^T \beta^t \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \lambda \left( \sum_1^T C_t - A \right)$$

$$\beta^t C_t^{-\sigma} - \lambda \leq 0 \text{ and } (\beta^t C_t^{-\sigma} - \lambda) C_t = 0$$

$$\sum_1^T C_t \leq A \text{ and } \left( \sum_1^T C_t - A \right) \lambda = 0$$

$$C_t = K \beta^{t/\sigma} = K \delta^t \quad (\delta = \beta^{1/\sigma}) \Rightarrow K = A \frac{1-\delta}{\delta} \frac{1}{1-\delta^T} = A \frac{1-\beta^{1/\sigma}}{\beta^{1/\sigma}} \frac{1}{1-\beta^{T/\sigma}}$$

$$\text{Special case: } T = 2 \Rightarrow C_1 = A \frac{1}{1+\beta^{1/\sigma}}; C_2 = A \frac{\beta}{1+\beta^{1/\sigma}}$$

$$\text{As } \sigma \rightarrow 0 \Rightarrow C_1 \rightarrow A, C_2 \rightarrow 0$$

$$\text{As } \sigma \rightarrow 1 \Rightarrow C_1 \rightarrow \frac{A}{1+\beta}, C_2 \rightarrow \frac{A\beta}{1+\beta}$$

$$\text{As } \sigma \rightarrow \infty \Rightarrow C_1 \rightarrow A/2, C_2 \rightarrow A/2$$

2. Why do we need to simulate?

# The concepts

- Maintained assumption: agents optimize objective functions subject to a set of constraints. These are: budget constraints, institutional constraints, ...
- State variables are taken as given by agents when making decisions (choosing values) on their control (decision) variables
- Time-t decision may become or determine time-(t+1) state variables: investment and capital stock
- Decision variables for some agents may be state variables for other agents: Public expenditures
- We need a closed system: as many optimality conditions as decision variables
- In the absence of uncertainty, linear-quadratic optimization problems
  - Quadratic objective functions, linear constraints
  - lead to a set of linear optimality (first-order) necessary conditions
  - Together with transversality conditions

2. Why do we need to simulate?

# Under uncertainty

- The system of first order conditions is no longer a closed system, since it involves expectations of functions of future decision and state variables. Those functions are linear in a linear-quadratic problem
- Need to make some assumption on the expectations formation mechanism: exogenous/endogenous
- We may face expectations at different horizons, based on different information sets
- Separation of control and estimation applies (certainty-equivalence principle): solve the deterministic control problem, and take expectations where needed afterwards
- Current values of decision variables depend on past values of decision and state variables and current expectations of future exogenous variables
- But not on current expectations of future decision variables, which are endogenous
- Solution methods for linear models: Whiteman (1983), Linear Rational Expectations Models, U. of Minnesota Press.

*2. Why do we need to simulate?*

# Partial equilibrium framework: Sargent (1979)

- Simple problem: a firm maximizes present value of current and expected future profits by choosing current employment on the basis of past state (wages) and decision variables (employment, investment) and current expectations of future state variables (real wages, productivity)
- If we add an assumption on expectations, we produce the optimal level of current decision variables
  - It is the perception of rational economic agents on the future evolution of exogenous variables that matters, rather than actual future values
  - Credibility of economic policy
  - Optimal decision rules for private agents depend on the structure and parameter values used in policy rules (Lucas' critique)

*2. Why do we need to simulate?*

# General equilibrium model

- Each agent solves a specific optimization problem
- Constraints and market clearing conditions are imposed on top of first order conditions for each agent's problem
- Equilibrium conditions should conceivably give us a closed system that we could solve for equilibrium paths of all endogenous variables
- But we enter a new level of difficulty:
  - We cannot assume any arbitrary perception about the stochastic process governing future prices
  - We rather need to simultaneously solve for agents' decision variables together with equilibrium prices
  - But if prices are endogenous, budget constraints are no longer linear  $\Rightarrow$  the certainty-equivalence principle no longer applies

*2. Why do we need to simulate?*

# Example: General equilibrium model

$$\text{Max} E_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$$

*subject to :*

$$K_{t+1} + C_t \leq \theta_t f(K_t) + (1 - \delta)K_t$$

$$C_t \geq 0, \quad K_{t+1} \geq 0$$

*given*  $K_0$  and  $\{\theta_t\}_{t=0}^{\infty}$

$$1 = \beta E_t \left[ \frac{U'(\theta_{t+1} f(K_{t+1})) + (1 - \delta)K_{t+1} - K_{t+2}}{U'(\theta_t f(K_t)) + (1 - \delta)K_t - K_{t+1}} (1 - \delta + \theta_{t+1} f(K_{t+1})) \right]$$

$$\lim_{t \rightarrow \infty} \beta^t E_t \lambda_t K_{t+1} = 0$$

*Special case :*

$$U(C) = \ln(C)$$

$$f(K) = K^\alpha, 0 < \alpha < 1$$

*Optimal decision rule :*

$$K_{t+1} = \alpha \beta \theta_t K_t^\alpha$$

- A typical consumer maximizes expected discounted value of current and future utility, with first order condition:

$$\frac{\partial U}{\partial C_t} = E_t \left[ (1 + r_t) \frac{\partial U}{\partial C_{t+1}} \right]$$

- How should this equation be handled?
- An analytical solution will exist only in a few special cases
- We need numerical simulations to obtain a large number of time series realizations for the main variables in the economy
- This is the only way to proceed
- But, whenever needed, it is an approximation method

*2. Why do we need to simulate?*

# Numerical solution methods

- Not specific to equilibrium models
    - Motivated by appearance of expectations of nonlinear functions of future state and decision variables
    - Together with endogenous expectations mechanisms (rationality)
    - They can accommodate frictions
  - When the equivalence between competitive equilibrium and central planning resource allocation mechanisms holds, it is helpful to obtain the solution to the competitive equilibrium model
    - But that does not mean that numerical solution methods are specific to situations where the second welfare theorem holds
  - Not restricted to representative agent economies
  - Not restricted to rational expectations models (limited rationality being an interesting alternative)
  - Not specific to Macroeconomics
  - Not specific of business cycle theories, where productivity shocks are the main source of fluctuations
- 2. Why do we need to simulate?*

3. What do we get out of  
model simulations?

- A theoretical, stochastic model can be seen as imposing a set of restrictions on the probability distribution of the vector stochastic process for the relevant variables
  - Analytical structure of the model
  - Parameter values
  - Multivariate probability distribution for the vector of exogenous shocks
- The solution to the model is the probability distribution of that vector stochastic process
- That cannot usually be characterized analytically
- Simulation methods provide us with an approximation: the frequency distribution for any characteristic of interest: impulse responses, a given regression coefficient, relative volatility, ...

*3. What do we get out of model simulations?*

- A numerical solution: a set of time series, one for each variable in the economy, satisfying all optimality conditions, budget constraints, and equilibrium conditions
- Model simulation is the procedure by which a numerical solution is computed
- And we compute a large number of solutions realizations
- Each of them can be used to compute the numerical value of any characteristic of the model
  - Leading to the sample frequency distribution
  - Estimation theory
  - Possibility of testing

*3. What do we get out of model simulations?*

# Steady-state

- Deterministic steady-state: constant rates of growth for per capita variables that can be maintained forever
  - Those constant growth rates may be zero in some economies
- Exogenous versus endogenous growth
  - Exogenous growth. Causes for growth: exogenous productivity growing at a constant rate
  - Endogenous growth. Causes for growth: tax rates, money growth, ...
- Stochastic steady-state: a set of time series fluctuating around a deterministic steady-state
- In exogenous growth models,
  - appropriately discounted variables remain constant in the deterministic steady-state
  - Fluctuate around constant reference values in a stochastic steady-state

*3. What do we get out of model simulations?*

- In endogenous growth models,
  - Per capita variables contain a unit root, even after discounting for any source of deterministic growth
- Usually, policy analysis was performed only on steady-state
  - A policy intervention would take the economy outside steady-state: from an old to a new steady-state
  - Long-run effects were characterized, and alternative policies compared on the basis of those effects under a given policy loss function
  - No reason why the analysis should be restricted to steady-state
  - Often, short-run effects run contrary to long-run effects
  - So that there is some compensation, being unclear which effect dominates
  - Important features: size of the policy effect, time needed to converge to the new steady-state (rather, to get close enough to it), utility discount factor
- Sample characteristics that can be obtained: sample means, standard deviations, relative volatility, linear correlation coefficients, impulse responses, regression coefficients (linear multipliers), VAR representations, decomposition of variance, spectral density matrices, VaR (value at risk)

*3. What do we get out of model simulations?*

- We can provide answers to probabilistic questions:
  - *What is the probability that such and such event would arise under each alternative policy considered?*
- The answer to which depends on:
  - The structure of the model economy
  - the chosen parameterization
  - the probability distribution assumed for the vector of exogenous shocks

*3. What do we get out of model simulations?*

# Interesting statistical analysis

- The standard deviation across the set of simulations of the marginal propensity to consume will not coincide with the one provided by the least-squares formula for a single realization
  - Which will change with each simulation
  - But will the average of estimated least-squares deviations for  $\beta$  be the same as the sample standard deviation (across the set of realizations for the numerical solution)?
  - The empirical distribution for a given characteristic does not need to be Gaussian even if innovations to exogenous stochastic processes are so. The model economy is a nonlinear transformation between two multivariate stochastic processes
  - The stochastic dimension of the solution cannot be larger than that of the stochastic inputs (although the model is not linear...)

*3. What do we get out of model simulations?*

## 4. Calibrating/estimating a theoretical model

# 4.1 What is a reasonable model?

- The question is not specific to simulation methods
- Simplicity versus realism
- Lucas (1980): Methods and problems of business cycle theories:
  - *“One of the functions of economic theory is to provide fully articulated artificial economic systems that can serve as laboratories in which policies that would be prohibitively expensive to experiment with in actual economies can be tested at a much lower cost...”*
  - *“Insistence in the realism of an economic model subverts its potential usefulness in thinking about reality. Any model that is well articulated to give clear answers to the questions we put to it will necessarily be artificial, abstract, patently unreal...”*

- [...] "...Not all well-articulated models will be equally useful. Though we are interested in models because we believe they may help us to understand matters about which we are currently ignorant, we need to test them as useful imitations of reality, by subjecting them to shocks for which we are fairly certain how actual economies would react. The more dimensions on which the model mimics the answers actual economies give to simple questions, the more we trust its answers to harder questions. This is the sense in which more realism in a model is clearly preferred to less".

- The researcher is not interested in verifying if the model is correct, since he/she knows from the beginning that it is not.
  - Should we test simple model economies and pretend we test general theories?
- A simple, stylized model may be able capture a number of characteristics in actual data
- Iterative method, from theory to data, and viceversa
- Once we converge, we need to evaluate our degree of satisfaction with the limit reached: the number and relevance of empirical aspects that the theoretical model can explain
- Robustness: how the answer to a question of interest changes with local or global changes in the structure of the model (functional forms, parameter values, probability distributions for exogenous shocks)

# Questions of interest:

- Is it possible to mimic a given empirical regularity using a particular model economy?
- Power function over a class of models
- How much of that regularity can be explained by impulses to a specific exogenous innovation?
- Is it possible to reduce the discrepancy between the implications from the theoretical model and actual data by adding a given feature into the model?
- How does the probability distribution for the endogenous vector stochastic process change after a change in the one for the exogenous vector?

# Examples:

- Can we use the neoclassical exogenous growth model to explain the empirical regularity, common to most countries, that the relative volatility of aggregate private consumption to output is less than 1?
- To what extent can we reproduce the values of those volatilities using only technology shocks?
- Does the goodness of fit of the model improve if we add shocks to preferences of individual consumers?
- What effect does it have on the economy to announce a given time path for tax rates on income for the next four years, versus the alternative of maintaining continuous discretionality on them?
- What is the effect on price volatility in the neoclassical, monetary growth model (Sidrauski (1967)) if the monetary authority quits the monetary rule of  $k\%$  annual growth for the money supply to implement a policy of controlling interest rates?
- Since the standard growth model predicts a high correlation between hours worked and productivity, which is contrary to empirical evidence, will incorporating a second sector, for human capital accumulation, improve upon this property of the model?

- Many limitations that, in a first analysis, are associated with the methodology of analyzing models through simulation are more apparent than real:
  - almost any model is able to mimic the value of an statistic of interest in an actual economy, just by appropriately picking parameter values,
    - Not always true, and nevertheless, that model might perform badly relative to other data characteristics. Use appropriate loss functions.
  - different models can be found that are able to account for the sample value of a given statistic and, yet, have radically different implications relative to evaluating alternative economic policies
    - lack of identification is, in many occasions, just a reflection of the fact that the loss function used to rank alternative models includes too short a list of arguments
    - different points in the parameter space able to produce a given data characteristic in a given model

## 4.2 What is calibration?

- Lack of a clear-cut definition
  - Associating numerical values to coefficients
  - Like estimation, but ...
  - Not a consequence of a formal estimation procedure
- Values for structural parameters are usually taken from estimates obtained with micro data: factor elasticities, degree of risk aversion, intertemporal elasticity of consumption
- Or from some casual empirical characteristics of the economy under study
- Some parameter values are chosen so that steady-state (nonlinear) expressions for some variables are equal to observed reference levels in actual data
- Discount factors chose to match real interest rates
- This comparison needs to take into account that time series data are often nonstationary

- A subset of parameters are determined this way, and the rest can be used to validate the model
  - Choose them so as to replicate some empirical features
  - But this sounds a lot like simulated General Method of Moments (SGMM) estimation that minimizes a loss function based on comparison of moments in actual and in simulated data
  - Sargent (1987): “...*correlations and cross-correlations can contain more information on a given model than sample averages...*”
  - Standard deviations of structural shocks are usually chosen so that the volatility of a key variable, such as output, or the ratios of volatilities of consumption, investment or hours worked to output, match that of a real economy.

## 4.3 Limitations in structural parameter calibration

- the range of values that can be considered for a key parameter such as the elasticity of intertemporal substitution is too wide for most purposes
- some empirical analysis may differ from others in potentially substantial aspects, so that choosing parameter estimates from any one of them is not fully consistent, and produces significant parameter uncertainty
- that some parameter values are chosen a priori implies a selection bias, since there are many empirical studies that could be used
- marginal propensity to consume estimated in a cross section and in time series data have different interpretation

## 4.4 Limitations in calibrating exogenous stochastic processes

- hard to find information from a real economy concerning the stochastic structure of technology shocks, shocks in preferences, errors of controlling money growth or tax revenues, or the correlations among them.
- in the simplest business cycle model, an AR(1) model is assumed for productivity shocks, so that the simulated output series exhibits a persistence similar to the GNP series in actual economies.
- extreme care must be used when calibrating models so as not to achieve a spurious adjustment to data through ad-hoc assumptions.
- exogenous perturbations are assumed to be independent when simulating. The ability of the economic authority to intentionally perturb policy variables, and establish some correlations between policy induced perturbations and observed exogenous shocks adds a new dimension to the analysis.

## 4.5 Calibration versus formal estimation

- From a Bayesian viewpoint, estimation is the solution to a problem of minimizing (the expected value of) a given loss function.
- Discussing their relative properties does not make much sense without a reference to the corresponding loss functions.
- Calibration, based in choosing values for some structural parameters as a function of long-run sample averages, can be interpreted to the light of a particular loss function
- Alternative estimation procedures: Generalized Method of Moments (GMM), Maximum Likelihood (MV), Simulated Method of Moments (SMM)

## 5. Incorporating parameter uncertainty / Evaluation procedures

# Statistical evaluation through a loss function

- Winter 1996 issue of the Journal of Economic Perspectives
- Advantages:
  - it avoids a possibly arbitrary election of parameter values,
  - it provides a measure of dispersion that can be used to evaluate the goodness of fit of model to data.
- Disadvantages,
  - it needs a specific selection of moments to be used in fitting evaluation,
  - there are finite sample biases, which may lead to spurious inference,
  - the type of uncertainty which is imposed on the model by an estimation process does not necessarily reflect the uncertainty faced by a researcher when calibrating a vector of parameters, which is specified more appropriately through Bayesian methods.

# A Bayesian approach to simulation

- Estimation by calibration is considered to be exact, disregarding the existence of uncertainty on parameter values
- This practice is too restrictive: precise numerical results are provided for a given set of selected statistics, but no measure of uncertainty is presented.
- A given belief on structural parameter values could be incorporated in the form of a prior probability distribution on the parameter space
- Consider actual data statistics as fixed numbers, while the uncertainty in simulated data is used to provide a measure of how well the model fits actual data. Rather than fixing their numerical values, empirical information on some parameters is used to build a probability distribution on the parameter space, and each simulation is computed with a different point drawn from that distribution

- *"The characteristics of a model reproduced in research must always come accompanied by indicators of the degree of uncertainty they embed, which is just a consequence of uncertainty on the right model specification, in the sense described by Leamer (1978). To adequately represent that uncertainty, it is necessary to incorporate uncertainty on parameter values directly in the simulation exercise"* [Canova and de Nicolo (1995)].
- After repeated simulations with different parameter vectors, we can compute either the size of the calibration tests, or the percentiles of the empirical distribution of the simulated statistic where the value of the statistic in actual data falls.
- Actual and simulated data are used symmetrically, and one could either ask whether actual data could be generated by the model or whether the simulated data is consistent with the distribution of the observed sample
- The confidence interval criterion, and the difference of means, proposed by DeJong, Ingram and Whiteman (1996) measure the degree of overlap of the distributions of actual statistics and those obtained from the model

# Evaluating procedures for simulation models

- Not even the need for model evaluation is uniformly accepted among calibrators.
- Interpreting calibration as estimation with no error forces an informal evaluation of the distance between actual and simulated statistics: once parameter values have been chosen, uncertainty comes only from the exogenous stochastic processes.
- The model establishes an exact relationship between endogenous variables and parameters, and we will not be able to conclude whether actual and simulated statistics are significantly different
- *"No attempt is made to determine the true model. All models are abstractions and are, by definition, false" [Kydland and Prescott (1982)].*
- *"...the trust a researcher has in an answer given by the model does not depend on a statistical measure of discrepancy, but on how much he believes in the economic theory used and in the measurement undertaken" [Kydland and Prescott (1991)].*

- If parameters are estimated with sampling error, then it makes sense to use measures of dispersion for simulated statistics that reflect parameter uncertainty.
- A quadratic distance between the vector of statistics computed from actual data and those obtained by simulation will asymptotically follow a chi-square distribution with degrees of freedom equal to the number of overidentifying restrictions.
- The weighing matrix in the quadratic form should be the variance-covariance matrix of the vector of statistics being used, which can easily be estimated  $\Rightarrow$  estimation by simulation methodology
- Specially interesting to compare statistics from the joint probability distribution of subsets of variables, like impulse response functions, VAR representations, cross-correlations or coherence functions
- Sources of dispersion in sample (simulated) statistics: i) parameter uncertainty, ii) sample variation in innovations, iii) model uncertainty, iv) numerical approximation in solution method

## 6. Stability

- Stability is crucial, but often neglected.
- The analysis of stability is different in a stochastic, dynamic model than in its deterministic version.
- It is also different in nature in endogenous than in exogenous growth models.
- Besides, with the exception of the special cases that lead to linear models, we will be facing the stability of a non-linear, possibly stochastic system, for which general analytical conditions are unknown.
- All we can do is build the best linear approximation to the model, and discuss stability in the approximated model.
- Using stability conditions makes an important difference in terms of the behavior of the paths generated as solutions to the model.

- Since the approximation must be made around some specific reference (usually the steady state of detrended variables in exogenous growth models), conclusions can only be local.
- Although such local analysis may be enough to characterize properties of fluctuations around steady state, it might not be enough when analyzing the effects of a policy intervention that takes the economy away from steady-state.
- Some numerical solution methods are better equipped than others to handle stability.
- Stability analysis provides useful information on the degree of determination of the model.
  - Local indeterminacy: more than one possible trajectory leading to the steady state.
  - Global indeterminacy, in that there is a multiplicity of steady states: Bubble equilibrium
  - Lack of solution

- Dealing with stability is not independent from dealing with expectations
- Stability gets harder in endogenous growth models.
  - The implied time series have a deterministic trend which can easily be taken care of, very much as in exogenous growth models,
  - but they also have a unit root.
  - Transitory shocks will have permanent effects even after detrending, i.e., after discounting for the deterministic steady state rate of growth.
  - So, there is a fundamental lack of stationarity which cannot be handled by just normalizing variables to produce time series realizations for ratios of variables, not for their levels.
  - But it is the levels of the relevant variables that are needed for welfare analysis of the kind used in policy evaluation

## 7. Some statistical issues

# Some statistical issues

- Non-parametric statistics is underutilized in Economics
- The extent to which a given statistic (relative variance of investment, capital/labor ratio, etc.) behaves similarly under different parameterizations or models  $\Rightarrow$  Not a problem of mean values
- Evaluating differences to the light of the comparison of the mean and variance of a given statistic across simulations, assumes the empirical distribution of the statistic to be Gaussian, when it is rather unlikely that Normality of the exogenous shocks will be preserved by a non-linear model
- Non-parametric tests even more interesting, since they are distribution-free.

- The convenience of using an average statistic to compare models is often questionable [use standard deviation appropriately]. But that distribution is often asymmetric, possibly multi modal.
- Much more appropriate: compute measures of distance between the empirical distributions emerging from two different models or two different parameterizations of a same model economy.
- Homogeneity tests between the (unknown) theoretical value of a given statistic under two parameterizations, or two different models, should be one-tailed. In most cases, there are theoretical reasons to believe that a given parameter vector, or feature of a model, or policy rule, will be more likely to account for a given stylized fact.
- A theoretical model should not be expected to produce a time series for output, say, that matches the actual pattern observed in the US economy.

- Most often, the adequacy of a model should be viewed as the degree to which the restrictions it introduces on the multivariate probability distribution of the set of relevant variables are also observed in the analogous probability distribution, estimated from actual data.
- Alternatively, Watson (1993) evaluates the goodness of fit of the model by computing how large a noise would have to be added to the implied time series, so that the same fluctuations as in actual data are obtained.
- We must be aware of the difference between these two approaches as well as the fact that, most often, we are in the former situation.
- Filtering numerical solutions is commonplace.
  - specially surprising when working with models which display (exogenous or endogenous) steady-state growth, since the average rate of growth produced by the model should be one of the important features to mimic or be taken into account when calibrating the model.
  - even the solution to models that display no growth is filtered when comparing with actual data.

# Numerical solutions open new possibilities for analysis

- Since the model can be simulated under any multivariate probability distribution for the set of exogenous shocks, we can characterize the properties of the model which are implied by each of the shocks impinging on the economy.
- It is enough to set variances for all other shocks as well as their correlations with the former shock, all equal to zero.
- We could find out that a given feature is due to a specific shock or, alternatively, that it can be produced by two of the exogenous shocks in the economy, although one of them produces it to a much lesser extent.
- What is true under a shock on preferences could be false when a single shock in productivity affects the economy.
- The consideration of demand versus supply shocks is one of the many interesting differences to be established.
- Policy analysis like Poole (1970), concluding what instrument the monetary authority should use as a function of what is the more important source of shocks in the economy, becomes a natural issue to discuss when numerical solutions are obtained.

- Do we change the type of questions we ask?
- We can provide not only a given parameterization that is able to explain a given stylized fact but, rather, a whole continuum of combinations of parameters with that ability.
  - Example: characterize the line representing combinations of variances for the productivity shock and the shock to money growth that can reproduce relative volatility of investment to output in US data.
- In fact, it is hard to believe that a given model parameterization which is able to reproduce a given stylized fact should be taken as a satisfactory answer.
- Produce the curve describing the functional relationship between the statistic considered and the value of each of the relevant parameters

- Models solved under assumption of rational expectations but the numerical solution is hardly ever tested for rationality.
- The solution time series allow for the realized value for the nonlinear function inside the conditional expectation to be computed each period, so that an expectation error can also be obtained
- The resulting time series for rational expectations errors should be autocorrelation-free, and uncorrelated with any variable in the information set at the time the expectation was formed [Den Haan and Marcet (1994)].

## 8. The analysis of the transition between steady states

# The analysis of the transition between steady states

- Without characterizing the transition, the only evaluations possible are those based on the levels that the relevant variables achieve in the respective steady-states, before and after the policy intervention.
- Analytical characterization of the temporal trajectories followed by the relevant variables in their transition towards steady-state is impossible
- Use them to aggregate over time the levels achieved by the policy-maker objective function
- Only numerical evaluation is possible
- So long as the empirical distributions (or the posterior distributions, in a Bayesian analysis) of the main statistics have low dispersion, we will be able to make precise statements on the general characteristics of the alternative transition processes

- Often, the welfare comparison of alternative policies tends to be contrary to the conclusion that would be achieved by focusing just on steady-state.
- As a consequence, the optimal choice of policy will finally rely on aspects such as:
  - the speed of convergence to steady-state,
  - the rate of time discount,
  - the concavity of preferences, and so on.
- Our conclusions may be different in economic structures that differ just in values for these parameters, even though initially, they were not directly related to the policy issue under discussion.
- Normative policy analysis, taking explicitly into account these crucial dynamic issues, is one of the most active research areas.

9. Have we changed the type of questions we ask?

- Maybe the most important contribution of numerical solution methods is that they have, in fact, changed the way we analyze models.
- We now ask new questions and compare models in dimensions that were not even considered a few years ago.
- Policy analysis is also viewed from a new, richer perspective, which explains that policy design is currently the most active research area in macroeconomics.
- Questions that relate to statistical characteristics of the model, like the volatility of a given variable, or cross-correlations between some variables, that would have been impossible to characterize analytically, even if such a solution would exist for the levels of the variables themselves.
- Some other times, the simulation process itself and the numerical observations it produces, suggest interesting model features that might have never shown up if an analytical answer was available.

*9. Have we changed the type of questions?*

# Should economic policy be cyclical or counter-cyclical?

- suppose that fluctuations in the expenditures/output ratio can be interpreted as controlled deviations around a pre-announced target level. Should they then be correlated with exogenous supply shocks?
- solving the model under different correlations and computing the frequency distribution of welfare
- we will have a welfare effect for each correlation level and each possible tax or combinations of tax rates (on consumption, capital or labor income, etc.) to be adjusted. This would have clear implications on the optimal way to conduct policy.
- this analysis makes sense even if we believe that the random deviation of the expenditures/output ratio is beyond the control of the economic authority, since there will still be a welfare-maximizing correlation.

*9. Have we changed the type of questions?*

- Using actual data, we could separately identify supply and fiscal shocks, possibly through an structural VAR type of analysis.
- Estimating the empirical correlation between both perturbations, and knowing the taxes which are most often being adjusted, we can figure out the extent to which that correlation is close to the level that the model predicts as optimum.
- Or, given the estimated correlation in actual data, interpreted now as a control error in public expenditures, we will be able to discuss which tax should be adjusted over the business cycle so that the Government budget constraint holds every period, and welfare is highest.

*9. Have we changed the type of questions?*

# Heterogeneous agents models

- Extremely interest for almost any issue in economics, from characterizing how markets work, to establishing a ranking among alternative economic policies:
  - to analyze the impact of possible liquidity constraints we want to consider a setup where some agents are restricted at a given period, while some other agents are not, adding to the model the conditions that define the flow between restricted and unrestricted agents from one period to the next.
- Something similar could be said about characterizing the effects of asymmetric information:
  - we then want to consider different types of agents, each one having access to different information sets, which might depend on their past investment on information. The distribution of information across agents then becomes endogenous.

*9. Have we changed the type of questions?*

- This increases complexity dramatically, since it implies following the time evolution of the distribution of the vector of state variables among all individual agents.
  - in an economy with a single cumulative good, in which agents differ in their initial endowment, it will be crucial to know the distribution of income across the population at the beginning of each decision period.
  - In an economy where the state variable is the productivity of each worker, the number of individuals with each level of productivity at a given period will be a key variable in determining prices and quantities. Agents with different levels of the state variables will follow a different behavior, so that the information on aggregate variables will not be enough to anticipate the evolution of the economy, being also necessary to know the distribution of the aggregate level of the state variable across the population.
- Calibrating these models requires an initial distribution, which must be chosen so that it matches the analogue distribution observed in actual data.

*9. Have we changed the type of questions?*

- The time evolution of the distribution of a state is a key factor determining the equilibrium dynamics, and a very important characteristic of the model.
- Computing an equilibrium will require finding a fixed point each period, not only in the space of prices, but also in the space of distributions of the state variables subject to heterogeneity: such distributions become additional states.
- To the standard conditions in representative agent models, consistency conditions among the optimality conditions of individual agents and the behavior of aggregate variables must be added when characterizing the competitive equilibrium [Rios-Rull (1995), Diaz-Gimenez (1997)].
- Heterogeneous agent models are being proposed to try to explain some empirical observations that cannot be explained in the representative agent framework, like the equity-premium puzzle.
- Besides, such an attempt opens again questions which are new in spirit to economic analysis, like the cyclical behavior of income distribution, the relevance of learning processes, and so on.

*9. Have we changed the type of questions?*

- Using these models is important because the answer to some traditional questions may turn out not to be robust to the consideration of heterogeneous agents economies: Imrohoroglu (1992)
- A promising use of these models in asset pricing can be seen in Marcet and Singleton (1999) and in Heaton and Lucas (1996).
- Rios-Rull has succeeded in approximating overlapping generation models to actual economies, considering that any single individual can live a large number of periods.
  - The impact of demographic changes in aspects like capital accumulation and social insurance can then be analyzed in an adequate framework.
  - Other business cycle issues like the volatility of hours worked, that we have mentioned at different points in this survey, can be nicely studied in such a model, and Rios-Rull (1992) has found different volatility across workers of different age as an equilibrium characteristic.
- How such a result mixes with the current structure of population to produce a specific result for volatility of aggregate hours is an illustration of the central point of this promising research agenda.

*9. Have we changed the type of questions?*

# The type of policy conclusions we reach

- Conclusions to policy analysis will often be of the sort:
  - *" ...if the main shocks in the economy are supply shocks, then it is better to implement a monetary policy aimed to maintain a given growth rate of money, while leaving interest rates to be determined in the market, the opposite is true if randomness enters mainly through the demand side"*
  - *" ...if the elasticity of intertemporal substitution is above a critical value, then it is better to adjust labor income taxes over the cycle while maintaining capital income taxes roughly stable, while the opposite is true if the elasticity of intertemporal substitution of consumption is below that value".*
- We are increasingly going to reach such contingent conclusions.

*9. Have we changed the type of questions?*

- Some researchers view such relativity as a weakness of economic analysis, suggesting that it would better to discuss policy in simpler models, even if they miss some interesting feature, since they allow for neater conclusions.
- The opposite is, however, more likely to be true.
- Economists may have been too ambitious in attempting to reach statements that would always be true, regardless of the type of economy being studied.
- In characterizing optimal policy as a function of the structure of the economy (source of shocks, structural parameter values, etc..) we are borrowing from Bayesian statisticians, who sometimes aim at providing their readers with a sort of catalogue, that defines the mapping between input (the structure of the economy) and output (the specification of optimal policy).
- Did we really believe that a similar kind of policy would be optimum for a variety of widely different economies and for any conceivable policy environment?

*9. Have we changed the type of questions?*

# 10. Summary

- Numerical solution methods are necessary
- Not specific to equilibrium economies
- Second welfare theorem is not needed
- The way they are used can still be improved:
  - Incorporate parameter uncertainty,
  - Estimation by simulation
  - Probabilistic model evaluation
  - Solving endogenous growth models
  - Rationality tests
- Huge possibilities (and limitations) in handling heterogeneity
- It has changed the type of questions we ask and it has made economic analysis much richer, and more explicitly contingent on assumptions

# References

- Heer, B., and Maussner, A., 2005, Dynamic general equilibrium modelling, Springer Verlag
- DeJong, D., and C. Dave, 2007, Structural Macroeconometrics, Princeton U. Press
- Canova, F, 2007, Methods for Applied Macroeconomic Research, Princeton U. Press
- Novales, A., E. Fernandez, J. Ruiz, 2007, Economic growth: Theory and numerical solution methods. Accessible at:  
[www.ucm.es/info/ecocuan/anc/#Growth](http://www.ucm.es/info/ecocuan/anc/#Growth)