Unit Roots and Cointegrating Matrix Estimation using Subspace Methods

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Outline

1. Introduction
   - Some previous works
   - Motivations
   - The framework
2. Main Contributions
   - Detection of Unit Roots (URs)
   - Cointegration analysis
3. Some Examples
   - Summary of the simulations
   - An example with real data
4. Conclusions

Cointegration, cointegrating rank and literature

Definition

Cointegration: When a vector of integrated variables share an equilibrium relation which turns out to be stationary

The literature provides many tools to deal with this feature:

- [Engle and Granger, 1987]
- [Johansen, 1988, Johansen, 1991]
- [Stock and Watson, 1988, Stock and Watson, 1993]
- [Poskitt, 2000]
- [Bauer and Wagner, 2002]

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Basic shortcoming and proposals

Shortcomings of the previous methods:

- Usually need the choice of lag length [Emerson, 2007]
- Most cointegration tests don't provide the empirical support that is expected [Bewley and Yang, 1998]
- Some require an initial "good" null hypothesis [Bauer and Wagner, 2002]

Our contribution:

- Leads to consistent estimates of UR
- Can be adapted to non-Gaussian distributions
- Provides a good insight into UR and cointegration relationships
- Also provides a consistent estimation of the cointegrating matrix

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The model

Consider a linear-fixed, non-explosive, strictly minimum-phase system in innovations form:

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{\Phi} \boldsymbol{x}_t + \boldsymbol{E} \psi_t \\ \boldsymbol{z}_t &= \boldsymbol{H} \boldsymbol{x}_t + \psi_t \end{aligned}$$

where $\boldsymbol{z}_t$ is a $m-$vector of (at most) I(1) series.

This system can be estimated by the CCA algorithm which also gives an estimation of the CCCs (σ_j).

Superconsistency : $\hat{\sigma}_j \rightarrow 1, j = 1, 2, ..., d$, faster than $\hat{\sigma}_j \rightarrow \sigma_j < 1, j = d + 1, ..., i$.

Determining the number of URs from the CCCs

From the idea of the *superconsistency* we propose:

$$URC(G_l) = 1 - \hat{\sigma}_j^2 - G_l(T, \bar{d}) \leq 0$$

proving that for some $G_l(T, \bar{d})$, $URC(G_l)$ consistently estimates the number of unit roots (d)

However, there are infinite $G_l(T, \bar{d})$ which give consistent estimates of d:

- How to choose (one or more) amongst all of them?

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Choosing amongst a collection of loss functions I

We select the penalty functions that fulfill (at least one of) the following conditions:

- *Condition 1:* The empirical size is 5% for $T = 50$.
- *Condition 2:* The empirical power is 50% for $T = 50$.

For instance, to test if there is one UR:

- Empirical size: Probability of rejecting a UR in a gaussian random walk: $\nabla z_t = a_t$
- Empirical (size-unadjusted) power: Probability of rejecting a UR in a persistent autoregressive process: $(1 - .9B)z_t = a_t$

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We select the penalty functions that fulfill (at least one of) the following conditions:

- *Condition 1:* The empirical size is 5% for $T = 50$ (G_a).
- *Condition 2:* The empirical power is 50% for $T = 50$, (G_b).

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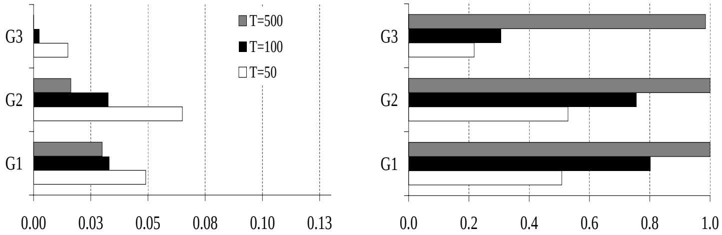
Choosing amongst a family of loss functions II

Empirical size and power of $URC(G_l)$ ($l = 1, 2, 3$) to test $d > 0$ in the process $\phi(B)z_t = a_t$:



Left plot: Empirical size when $\phi(B) = 1 - B$.
Right plot: Empirical power when $\phi(B) = 1 - .9B$

$G1$ will become G_a and G_b (just in this particular case!)

Estimating the cointegrating rank

When working with multiple I(1) series, we propose a procedure to determine the cointegrating rank (c) which consists of:

1. Checking that the series are I(1), applying the criterion $URC(G_l)$ ($l = a, b$) to every single series.
2. Estimating the number of unit roots (d) of the multivariate process.
3. Estimating the cointegrating rank as the difference between the system dimension and the number of unit roots estimated in the previous step: $\hat{c} = m - \hat{d}$.

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Estimating the cointegrating matrix

If $\boldsymbol{Z}_t$ is made up of I(1) series and we have estimated the cointegrating rank (c) then:

$$\begin{pmatrix} \boldsymbol{X}_{1,t+1} \\ \boldsymbol{X}_{2,t+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_d & \boldsymbol{0} \\ \boldsymbol{0} & \phi_{n-d} \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_{1,t} \\ \boldsymbol{X}_{2,t} \end{pmatrix} + \begin{pmatrix} \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \end{pmatrix} \psi_t$$
$$\begin{pmatrix} \boldsymbol{Z}_{1,t} \\ \boldsymbol{Z}_{2,t} \end{pmatrix} = \begin{pmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{X}_{1,t} \\ \boldsymbol{X}_{2,t} \end{pmatrix} + \begin{pmatrix} \psi_{1,t} \\ \psi_{2,t} \end{pmatrix}$$

Premultiplying the observation equation by $\boldsymbol{\Lambda} = (\boldsymbol{I}_c \quad -\boldsymbol{H}_{11}\boldsymbol{H}_{21}^{-1})$:

$$\boldsymbol{Z}_{1,t} - \boldsymbol{H}_{11}\boldsymbol{H}_{21}^{-1}\boldsymbol{Z}_{2,t} = (\boldsymbol{H}_{12} - \boldsymbol{H}_{11}\boldsymbol{H}_{21}^{-1}\boldsymbol{H}_{22})\boldsymbol{X}_{2,t} + \psi_{1,t} - \boldsymbol{H}_{11}\boldsymbol{H}_{21}^{-1}\psi_{2,t}$$

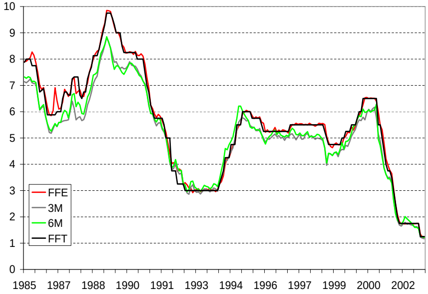
which is a stationary linear combination of $\boldsymbol{Z}_t$.
So $\hat{\boldsymbol{\Lambda}} = (\boldsymbol{I}_c \quad -\hat{\boldsymbol{H}}_{11}\hat{\boldsymbol{H}}_{21}^{-1})$ is a consistent estimate of the cointegrating matrix if $\hat{\boldsymbol{H}}$ is also consistent.

- DGP used by Riemers (1992), Toda (1995) or Poskitt (2000).
- Smallest probability of correctly estimating c is presented.
- Values are means for different sample sizes.
- A measure of reliability (which of them performs less badly)

	$URC(G_a)$	$URC(G_b)$	∇_T	LR_T	PLR_T^{SBC}	PLR_T^{LP}
			6 Bivariate systems			
mean	0.0307	0.0115	0.0	0.4012	0.0173	0.2110
rank	3	5	6	1	4	2
			5 Trivariate systems			
mean	0.2003	0.6837	0.0787	0.1973	0.1395	0.0868
rank	2	1	6	3	4	5
			4 Pentavariate systems			
mean	0.4965	0.8403	0.5813	0.0150	-	0.0768
rank	3	1	2	5	-	4

How does the methodology perform with real data?

The data comes from the Federal Reserve Bank of St. Louis



$URC(G_a)$ and $URC(G_b)$ determine 1 UR in each single series ...
... and 1 UR in the whole process.

$$\hat{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & \underset{(.053)}{-1.01} \\ 0 & 1 & 0 & \underset{(.082)}{-0.92} \\ 0 & 0 & 1 & \underset{(.071)}{-0.93} \end{pmatrix}$$

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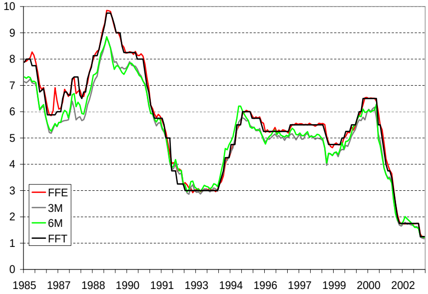

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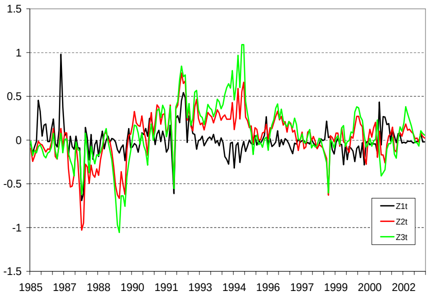
USA short term interest rates



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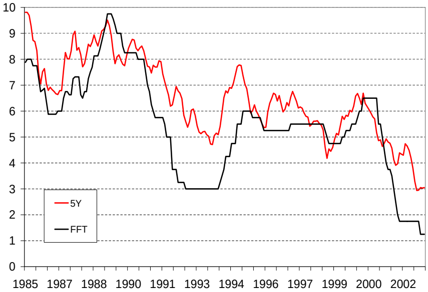
$$
\begin{aligned}
Z_{1t} &= FFE_t - 1.01 FFT_t \\
Z_{2t} &= 3m_t - 0.92 FFT_t \\
Z_{3t} &= 6m_t - 0.93 FFT_t
\end{aligned}
$$

are I(0).

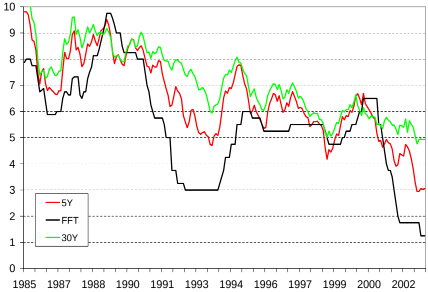
USA long term interest rates



$URC(G_a)$ and $URC(G_b)$ determine 1 UR in each single series ...
... and 2 UR in the whole process.

Then no cointegration relationship is found!

USA long term interest rates



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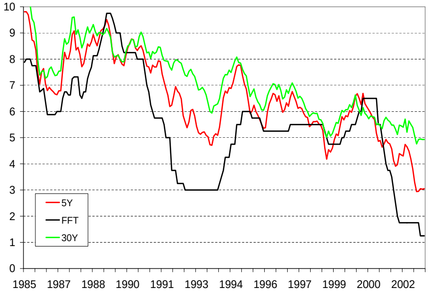

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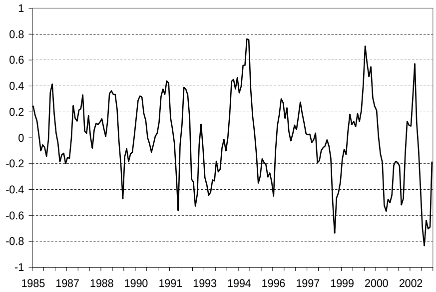

$URC(G_a)$ and $URC(G_b)$ determine 1 UR in each single series ...

... and 1 UR in the whole process.

$$Z_t = Y_{5,t} - 0.68 Y_{30,t} - 0.30 FFT_t$$

is I(0).

Summary

We propose:

1. a family of information criteria to estimate the number of UR (univariate and multivariate processes),
2. which can also be used to estimate the cointegrating rank (in I(1) series processes),
3. a consistent estimator of the cointegrating matrix.

The simulations show that:

- the methodology has a remarkable capacity of estimating the cointegrating rank ...
- ... which is more substantial in processes of high dimension.

And finally ...

... THANK YOU FOR YOUR TIME!

For Further Reading I

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