ABSTRACT INTERPOLATION THEORY AND SOME OF ITS APPLICATIONS IN ANALYSIS

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$$L_{p_1}(U, d\mu) \longrightarrow L_{q_1}(V, d\nu) \quad \text{with norm} \quad M_1 .$$

Take any $0 < \theta < 1$ and put $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$. Then T maps continuously

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▷ Marcel Riesz (1926); G.O. Thorin (1938)

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$$T : L_1([0,2\pi]) \longrightarrow \ell_\infty \quad \text{is bounded with norm} \quad M_0 \le \frac{1}{2\pi}$$

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For $1 < p < 2$, if we choose $0 < \theta < 1$ such that $1/p = (1-\theta)/1 + \theta/2$ and we put p' for the conjugate index of p, $1/p + 1/p' = 1$. Then $1/p' = \theta/2 = (1-\theta)/\infty + \theta/2$.

$$T : L_p([0,2\pi]) \longrightarrow \ell_{p'} \text{ is bounded with norm } M \leq \Big(\frac{1}{2\pi}\Big)^{1-\theta}\Big(\frac{1}{\sqrt{2\pi}}\Big)^{\theta} = \Big(\frac{1}{2\pi}\Big)^{1/p}.$$

Hausdorff-Young inequality.- If $1 < p < 2$ and $f \in L_p([0, 2\pi])$ then $(\hat{f}(m)) \in \ell_{p'}$ and

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Suppose, in addition, that

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Problem.- $(L_{p_0}, L_{p_1}) \rightsquigarrow (A_0, A_1)$, $(L_{q_0}, L_{q_1}) \rightsquigarrow (B_0, B_1)$, $L_p \rightsquigarrow A$, $L_q \rightsquigarrow B$.

Abstract interpolation theory

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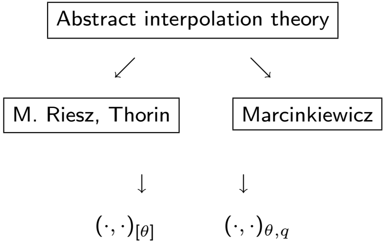

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- Function Spaces.
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- Harmonic Analysis.
- Partial Differential Equations.

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▷ J.L.-Lions and J. Peetre (Inst. Hautes Études Sci. Publ. Math. 19 (1964) 5-68)

For $0 < \theta < 1$ and $1 \le q \le \infty$, the *real interpolation space* $(A_0, A_1)_{\theta,q}$ consists of all $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\theta,q} = \left(\int_0^\infty \left(t^{-\theta} K(t,a) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \text{ if } 1 \le q < \infty \,,$$

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It turns out that

$$A_0 \cap A_1 \hookrightarrow (A_0, A_1)_{\theta,1} \hookrightarrow [A_0, A_1]_\theta \hookrightarrow (A_0, A_1)_{\theta,\infty} \hookrightarrow A_0 + A_1$$

Moreover, these constructions behave well with respect to bounded linear operators.

Let $\bar{B} = (B_0, B_1)$ be another Banach couple. If T is a linear operator from $A_0 + A_1$ into $B_0 + B_1$, whose restrictions $T : A_j \longrightarrow B_j$ are bounded for $j = 0, 1$, then we write $T \in \mathcal{L}(\bar{A}, \bar{B})$.

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Interpolation theorem.- Let $0 < \theta < 1, 1 \leq q \leq \infty$ and let $\mathfrak{F}$ be the real method $(\cdot, \cdot)_{\theta,q}$ or the complex method $[\cdot, \cdot]_\theta$. Given any Banach couples $\bar{A} = (A_0, A_1), \bar{B} = (B_0, B_1)$ and any operator $T \in \mathcal{L}(\bar{A}, \bar{B})$, the restriction of T to $\mathfrak{F}(A_0, A_1)$ is a bounded operator $T : \mathfrak{F}(A_0, A_1) \longrightarrow \mathfrak{F}(B_0, B_1)$ with norm

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For $f \in L_1(U) + L_\infty(U)$, we have

$$K(t, f; L_1(U), L_\infty(U)) = \int_0^t f^*(s)ds \quad , \quad t > 0.$$

Here f^* is the *non-increasing rearrangement* of f on $(0, \infty)$ is given by

$$f^*(t) = \inf\{\delta > 0 : \mu(\{x \in U : |f(x)| > \delta\}) \leq t\} .$$

Theorem .- If $1 \leq q \leq \infty, 0 < \theta < 1$ and $1/p = 1 - \theta$, then we have with equivalence of norms

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For $1 \leq p < \infty$, the Schatten p-class is defined by

$$S_p = \{T \in S_\infty : \|T\|_{S_p} = \Big(\sum_{n=1}^{\infty} s_n(T)^p\Big)^{1/p} < \infty\}$$

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Theorem.- Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples and let $T \in \mathcal{L}(\bar{A}, \bar{B})$ such that $T : A_0 \longrightarrow B_0$ is compact. For any $0 < \theta < 1$ and $1 \leq q \leq \infty$, we have that

$$T : (A_0, A_1)_{\theta,q} \longrightarrow (B_0, B_1)_{\theta,q}$$

is compact.

Proposition.- Let $0 < \theta < 1, 1 \leq q \leq \infty$ and let $\mathfrak{F}$ be the real method $(\cdot,\cdot)_{\theta,q}$ or the complex method $[\cdot,\cdot]_\theta$. Given any Banach couple $\bar{A} = (A_0, A_1)$, we have

(1)
$$\|a\|_{\mathfrak{F}(\bar{A})} \leq C \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^{\theta} \quad , \quad a \in A_0 \cap A_1 \, .$$

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$$K(t,a) \leq C t^{\theta} \|a\|_{\mathfrak{F}(\bar{A})} \ , \ a \in \mathfrak{F}(\bar{A}) \ , \ t > 0 \, .$$

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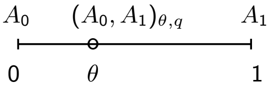

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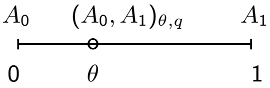

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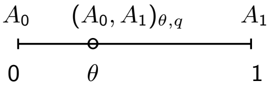

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▷ Cohen, DeVore, Petrushev and Xu (Amer. J. Math. 121 (1999) 587-628)

▷ Bechler, DeVore, Kamot, Petrova and Wojtaszczyk (Trans. AMS 359 (2007) 619-635)

▷ Cobos and Kruglyak (preprint 2009)

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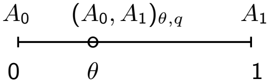

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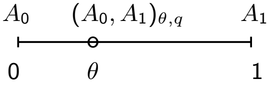

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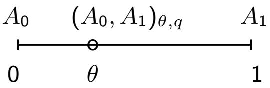

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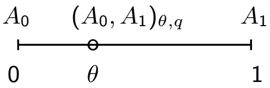

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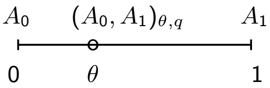

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• Recent results for $\theta = 0, 1$

▷ Cobos, Fernández-Cabrera, Kühn and Ullrich (J. Funct. Anal. 2009)