# Time-frequency Analysis meets Complex Analysis

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# Group of Complex and Harmonic Analysis at the Department of Mathematical Sciences, NTNU

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- Geometrical function theory
- Continuous fractions, Moment problems
- Spectral theory
- Applications to signal analysis

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Mathematical didactic

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- This is one of my favorite subjects;
- Time-frequency analysis (TFA) is an intensively developed subject which links topics in physics and engineering with a variety of mathematical subjects
- In this topic we have many common interests with our Spanish colleagues.

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- Further results

# Time Frequency Analysis

Setting of the problem:

*Given a signal $f(t)$ find which frequency (frequencies) does it have at each single moment $t$ of time and write down the corresponding representation*

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Yet each of you knows the answer:

# Music score


Free sheetmusic from www.8notes.com
Für Elise
Moderato
Ludwig van Beethoven (1770-1827)
p
mf

# The task of TFA:

UNIVERSAL MATHEMATICAL MODEL OF MUSIC SCORE

# Single note

Take a function $g$ such that both $g$ and $\hat{g}$ were well concentrated around the origin, meaning

$$\Delta(g) = \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt \int_{-\infty}^{\infty} \omega^2 |\hat{g}|^2 d\omega$$

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were small. For example $g(t) = e^{-t^2/2}$.
Given $t_0$ and $\omega_0$ the function

$$g_{t_0,\omega_0}(t) = e^{2\pi i \omega_0 t} g(t - t_0)$$

represents a "single note" concentrated around $t_0$ and time and around $\omega_0$ in frequency.

# Mathematical musical score:

can be viewed as follows:

Fix a "generator" $g$ and $a, b > 0$ and consider the expansion *(Gabor series)*

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Q : When such expansion exists and possesses "good properties"?
"Good properties" = There are constants $0 < c < C < \infty$ such that *frame property* holds

$$c \int_{-\infty}^{\infty} |f(t)|^2 dt < \sum_{m,n} |c_{m,n}|^2 < C \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

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Seip (1992), L. (1992) each $ab < 1$ provides the frame property !
Techniques: heavy complex analysis !
Idea of density : if we know a holomorphic function in a sufficiently dense set of points, we can reconstruct/ estimate it.

# Why complex analysis ?

There is a Fock space of entire functions:

$$\mathcal{F} = \{F \ \text{ analytic in the whole complex plane } \mathbb{C} \text{ and}$$
$$\|F\|^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dxdy < \infty\}.$$

<u>Fact</u> Frame property for the system $\{e^{2\pi iant} g(t - bn)\}_{n,m=-\infty}^{\infty} \Leftrightarrow$ *Sampling property* for the lattice $\{am + ibn\}_{m,n=-\infty}^{\infty}$ in $\mathcal{F}$, i.e. for some $0 < c, C < \infty$

$$c\|F\|^2 < \sum |F(am + ibn)|^2 e^{-\pi(a^2m^2+b^2n^2)} < C\|F\|^2, \ F \in \mathcal{F}.$$

Sampling theory for Fock type spaced of entire functions (Ortega-Cerda/Seip, Massaneda/Marko/Ortega-Cerda)
Ortega-Cerda/Seip (2002): Description of all exponential frames in $L^2(-\pi, \pi)$.

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Numerical results: Strohmer (Univ. of Californa, Davis).
Precise result (Borichev, Groechenig, L (2009):

$$\kappa(ab) \sim \frac{1}{1 - ab}.$$

Techniques ? Again complex analysis : atomization techniques i.e. approximation of a potential of some measure by a discrete potential (L./ Malinnikova, 2001), ( Ortega-Cerda, 2002)

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Perhaps something is wrong with complex analysis ?

# Yes !

Ascenzi, Bruna (2006): The only generator which leads one to space of holomorphic function is the Gaussian $g(t) = e^{-t^2/2}$.

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$$\langle f, e^{2\pi i \mu_n t} g(t - \nu_n) \rangle = 0 \Rightarrow f = 0$$

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Type of problem: uniqueness and sampling theorems in spaces with reproducing kernels.

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The corresponding problem for functions from the Fock space involves expression of the form

$$F'(\lambda_{m,n}) - \pi\bar{\lambda}_{m,n}F(\lambda_{m,n}),\ \lambda_{m,n} = am + ibn$$

instead of just $F(\lambda_{m,n})$.
This changes the situation drastically !

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